Journal of Applied Data Analysis and Modern Stochastic Modelling
Vol. 1 No. 1 (June, 2024)

Received: 16th January 2024 *Revised: 10th March 2024* *Accepted: 25th May 2024*

ON COMPLETENESS OF FLOW GENERATED BY A STOCHASTIC ALGEBRAIC-DIFFERENTIAL EQUATION WITH FORWARD MEAN DERIVATIVE SATIFYING THE RANK-DEGREE CONDITION

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ABSTRACT. We find conditions under which all solutions of stochastic algebraic-differential equations given in terms of forward Nelson's mean derivatives, exist for all $t \in [0, \infty)$. We suppose that the matrix pencil of equation satisfies the rank-degree condition.

Introduction

The notion of mean derivatives (forward, backward, symmetric and antisymmetric) was introduced by E. Nelson in [1, 2, 3]. In [4] (see also [5] where all preliminaries about mean derivatives are given) an additional mean derivative, called quadratic, was introduced so that from some Nelson's mean derivative and the quadratic one it became in principle possible to find a stochastic process having those derivatives.

In this paper we investigate the completeness property of the flows generated by the stochastic algebraic-differential equations given in terms of forward Nelson's mean derivatives, i.e., we find conditions, under which all solutions of those equations exist for all $t \in [0, \infty)$. Previously, in [6], this problem was investigated for equations given in terms of symmetric mean derivatives. The case of forward mean derivatives requires absolutely different methods of investigation. We suppose that the matrix pencil of equation satisfies the rank-degree condition. This assumption makes the investigation more successful.

The structure of the paper is as follows. In Section 1 we give some facts from the theory of matrices, necessary for the description of algebraic-differential equations. Section 2 is devoted to preliminaries of the theory of mean derivatives. In Section we present the main results of the paper.

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Date: Date of Submission January 8, 2024 ; Date of Acceptance February 05, 2024 , Communicated by Mamadsho Ilolov.

2010 *Mathematics Subject Classification.* Primary Primary 60H10; Secondary 60H30.

Key words and phrases. Mean derivatives; stochastic algebraic-differential equations; global in time solution.

* The work is supported by RSCF Grant No. 24-21-00004.

1. Some facts from the theory of matrices

Everywhere below we deal with the n dimensional linear space $\mathbb{R}^n$, vectors from $\mathbb{R}^n$ and $n \times n$ matrices.

Consider two $n \times n$ constant matrices A and B where A is degenerate while B is non-degenerate. The expression $\lambda A + B$, where λ is real parameter, is called the matrix pencil. The polynomial $\theta(\lambda) = \det(\lambda A + B)$ is called the characteristic polynomial of the pencil $\lambda A + B$. The pencil is called regular, if its characteristic polynomial is not identically equal to zero.

If the matrix pencil $\lambda A + B$ is regular, there exist to non-degenerate linear operators P (acts from the left side) and Q (acts from the right side) that reduce the matrices A and B to the canonical quasi-diagonal form (see [7]). In the canonical quasi-diagonal form, under appropriate numeration of basis vectors, in the matrix PAQ first along diagonal there is the $d \times d$ unit matrix and then along diagonal there are the Jordan boxes with zeros on diagonal. In PBQ in the lines corresponding to Jordan boxes, there is the unit matrix, and in the lines corresponding to the unit matrix there is a certain non-degenerate matrix J. Thus

$$P(\lambda A(t) + B(t))Q = \lambda \begin{pmatrix} I_d & 0 \\ 0 & N(t) \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}, \tag{1.1}$$

The non-degenerate pencil satisfies the rank-degree condition if

$$rank(A(t)) = \deg(\det(\lambda A(t) + B(t))). \tag{1.2}$$

If the pencil satisfies the rank-degree condition, formula (1.1) takes the form

$$P(t)(\lambda A(t) + B(t))Q(t) = \lambda \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}. \tag{1.3}$$

where J is non-degenerate since B is non-degenerate.

Consider a symmetric positive definite (i.e. non-degenerate) $d \times d$ matrix $\Xi(t)$.

Lemma 1.1. *(*[4, Lemma 2.2]*, see also* [5]*) There exists a $d \times d$ matrix $A(t)$ such that $\Xi(t) = AA^*$ where A^* is the transposed matrix A.*

2. Mean derivatives

In this section we briefly describe preliminary facts about mean derivatives. See details in [1, 2, 3, 5].

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, T]$, given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is an L_1 random element for all t. It is known that such a process determines 3 families of σ-subalgebras of the σ-algebra $\mathcal{F}$:

(i) "the past" $\mathcal{P}_t^\xi$ generated by preimages of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$;

(ii) "the future" $\mathcal{F}_t^\xi$ generated by preimages of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \le s \le T$;

(iii) "the present" ("now") $\mathcal{N}_t^\xi$ generated by preimages of Borel sets from $\mathbb{R}^n$ under the mapping $\xi(t) : \Omega \to \mathbb{R}^n$.

All the above families we suppose to be complete, i.e., containing all sets of probability zero.

For the sake of convenience we denote by E^{ξ}_t the conditional expectation $E(\cdot|\mathcal{N}^{\xi}_t)$ with respect to the "present" $\mathcal{N}^{\xi}_t$ for $\xi(t)$.

Following [1, 2, 3], introduce the following notions of forward mean derivatives.

Definition 2.1. The forward mean derivative $D\xi(t)$ of $\xi(t)$ at the time instant t is an L_1 random element of the form

$$D\xi(t) = \lim_{\triangle t\to +0} E^{\xi}_t\Big(\frac{\xi(t+\triangle t)-\xi(t)}{\triangle t}\Big), \tag{2.1}$$

where the limit is supposed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\triangle t \to +0$ means that $\triangle t$ tends to 0 and $\triangle t > 0$.

Remark 2.2. If $\xi(t)$ is a Markov process then evidently E^{ξ}_t can be replaced by $E(\cdot|\mathcal{P}^{\xi}_t)$ in (2.1) and by $E(\cdot|\mathcal{F}^{\xi}_t)$ in (**??**). In initial Nelson's works there were two versions of definition of mean derivatives: as in our Definition **??** and with conditional expectations with respect to "past" and "future" as above that coincide for Markov processes. We shall not suppose $\xi(t)$ to be a Markov process and give the definition with conditional expectation with respect to "present" taking into account the physical principle of locality: the derivative should be determined by the present state of the system, not by its past or future.

One can easily derive that for an Ito process $\xi(t) = \int_0^t a(s)ds + \int_0^t A(s)dw(s)$ its forward mean derivative takes the form $D\xi(t) = a(t)$ since $\int_0^t A(s)dw(s)$ is a martingale and so $D\int_0^t A(s)dw(s) = 0$.

Following [4] (see also [5]) we introduce the differential operator D_2 that differentiates an L_1 random process $\xi(t)$, $t \in [0,T]$ according to the rule

$$D_2\xi(t) = \lim_{\triangle t\to +0} E^{\xi}_t\Big(\frac{(\xi(t+\triangle t)-\xi(t))(\xi(t+\triangle t)-\xi(t))^*}{\triangle t}\Big), \tag{2.2}$$

where $(\xi(t+\triangle t)-\xi(t))$ is considered as a column vector (vector in $\mathbb{R}^n$), $(\xi(t+\triangle t)-\xi(t))^*$ is a row vector (transposed, or conjugate vector) and the limit is supposed to exists in $L_1(\Omega, \mathcal{F}, \mathsf{P})$. We emphasize that the matrix product of a column on the left and a row on the right is a matrix. It is shown that $D_2\xi(t)$ takes values in $\bar{S}_+(n)$, the set of symmetric semi-positive definite matrices. We call D_2 the quadratic mean derivative.

One can easily derive that for an Ito process $\xi(t) = \int_0^t a(s)ds + \int_0^t A(s)dw(s)$ its quadratic mean derivative takes the form $D_2\xi(t) = AA^*$ (see [4] and also [5]).

Remark 2.3. From the properties of conditional expectation (see, e.g., [8]) it follows that there exist Borel mappings $a(t,x)$, $a_*(t,x)$ and $\alpha(t,x)$ from $R\times\mathbb{R}^n$ to $\mathbb{R}^n$ and to $\bar{\mathrm{S}}_+$, respectively, such that $D\xi(t) = a(t,\xi(t))$, $D_*\xi(t) = a_*(t,\xi(t))$ and $D_2\xi(t) = \alpha(t,\xi(t))$. Following [8] we call $a(t,x)$, $a_*(t,x)$ and $\alpha(t,x)$ the regressions.

3. The main result

Let $\Xi(t), t \in [0,\infty)$ be a continuous symmetric positive definite (i.e. non-degenerate) $d\times d$ matrix. By Lemma 1.1 there exists $d\times d$ matrix A such that

$\Xi(t) = A(t)A^*(t)$. Construct the $n \times n$ matrix Θ of the form

$$\Theta = \begin{pmatrix} \Xi(t) & 0 \\ 0 & 0 \end{pmatrix} \tag{3.1}$$

We investigate the following stochastic algebraic-differential system

$$\begin{cases} LD\eta(t) = M\eta(t) + f(t) \\ D_2\eta(t) = \Theta \end{cases} \tag{3.2}$$

where L and M are from formula (1.3) and $f(t)$ is a smooth deterministic vector in $\mathbb{R}^n$. Taking into account the struicture of matrices L and M we see that system (3.2) is decomposed into to independent systems — the one in upper left corner with the unit matrix in L and matrix J in M and the system in right bottom corner with 0 in L and and the unit matrix in M. Let the unit matrix in L and the matrix J in M be $d \times d$ matrices, then the init matrix in the right bottom corner in M is a $(n-d) \times (n-d)$ matrix.

The system in upper left corner takes the form

$$\begin{cases} D\eta_{(1)} = J\eta_{(1)} + f_{(1)} \\ D_2\eta_{(1)} = \Xi \end{cases} \tag{3.3}$$

wshere $\eta_{(1)}$ and $f_{(1)}$ are constructed from the first d coordinates of the vectors $\eta(t)$ and $f(t)$, respectively.

The bloc in the right bottom corner takes the form

$$\begin{cases} 0 = \eta_{(2)} + f_{(2)} \\ D_2\eta_{(2)} = 0 \end{cases} \tag{3.4}$$

where $\eta_{(2)}$ and $f_{(2)}$ are constructed from the last $n-d$ coordinates of vectors $\eta(t)$ and $f(t)$, respectively.

It is evident that solution of (3.2) exists for $t \in [0,\infty)$ if and only if the same is valid for solutions of (3.3) and of (3.4). We will start with (3.4).

Theorem 3.1. *The process $\eta_{(2)}$, the solution of* (3.4)*, is deterministic and exists for all $t \in [0,\infty)$.*

Proof. Since $D_2\eta_{(2)} = 0$, the process $\eta_{(2)}$ is deterministic. From the first line of (3.4) we obtain that $\eta_{(2)} = -f_{(2)}(t)$ that exists for $t \in [0,\infty)$ be definition. □

Hence the completeness of the flow generated by (3.2) in $\mathbb{R}^n$ depends only on the completeness of the flow generated by (3.3) in $\mathbb{R}^d$.

Now we turn to (3.3). Here we will find several conditions under which the flow, generated by (3.3), is complete, i.e., the solution of (3.3) exist for $t \in [0,\infty)$.

Definition 3.2. The flow $\xi(s)$ is complete on $[0,T]$ if every orbit $\xi_{t,m}(s)$ a.s. exists for any couple (t,x) (with $0 \le t \le T$) and for all $s \in [t,T]$. The flow $\xi(s)$ is complete if it is complete on any interval $[0,T] \subset \mathbb{R}$.

The structure of equation (3.3) means that its solution satisfies the following stochastic differential equation in Ito form

$$\eta_{(1)}(t) = \int_0^t J\eta_{(1)}(s)ds + \int_0^t f_{(1)}(s)ds + \int_0^t Adw(s) \tag{3.5}$$

where A is such that $AA^* = \Xi$ (see above).

Hence the backward equation takes the form

$$\widehat{\eta}(t) = -\int_0^t J\widehat{\eta}_{(1)}(s)ds - \int_0^t f_{(1)}(s)ds + \int_0^t tr\, A'(A)ds - \int_0^t Adw(s) \qquad (3.6)$$

Denote by $\mathcal{A}$ and by $\hat{\mathcal{A}}$ the generator of flow generated by equation (3.5) and the backward generator, respectively.

Definition 3.3. A function from a topological space X to the real line $\mathbb{R}$ is called proper if the preimage of every relatively compact set in $\mathbb{R}$ is relatively compact in X.

Theorem 3.4. *Let there exist a smooth proper function φ on $\mathbb{R}^n$ such that $\mathcal{A}(t,x)\varphi < C$ for some $C > 0$ at all $t \in [0,+\infty)$ and $x \in \mathcal{R}^n$ where $\mathcal{A}(t,x)$ is the generator of flow $\xi(s)$. Then the flow $\xi(t,s)$ is complete.*

Theorem 3.4 is a simple version of rather general sufficient condition [9, Theorem IX. 6A].

Corollary 3.5. *On $\mathbb{R} \times \mathbb{R}^n$ consider the flow $\widetilde{\xi}(s) = (s, \xi(s))$ with the generator $\widetilde{\mathcal{A}}(t,x) = \frac{\partial}{\partial t} + \mathcal{A}(t,x)$. Let on $\mathbb{R} \times \mathbb{R}^n$ there exist a proper function $\widetilde{\varphi}$ such that $\widetilde{\mathcal{A}}(t,x)\widetilde{\varphi} < C$ for some $C > 0$ at all $t \in [0,+\infty)$ and $x \in \mathcal{R}^n$. Then the flow $\xi(s)$ on $\mathbb{R}^n$ is complete.*

Definition 3.6. We say that the flow $\xi(s)$ is continuous at infinity if for any finite interval $[0,T] \subset \mathbb{R}$, any $0 \le t \le T$ and any compact $K \subset M$ the equality

$$\lim_{x\to\infty} \mathrm{P}(\xi_{t,x}(T)) \in K) = 0 \qquad (3.7)$$

holds where $\xi_{t,x}(s)$ is the orbit of flow $\xi(s)$ such that $\xi_{t,x}(t) = x$.

Let the flow $\xi(s)$ generated by equation (3.5) be a flow of diffeomorphisms, i.e., the backward flow exists.

Theorem 3.7. *The forward flow $\xi(s)$ and the backward flow $\widehat{\xi}(s)$ generated by equation* (3.3)*, are simultaneously both complete and continuous at infinity if and only if on $\mathbb{R} \times \mathbb{R}^n$ there exist positive smooth proper functions $u(t,x)$ and $\widehat{u}(t,x)$ such that the inequalities*

$$\left(\frac{\partial}{\partial t} + \mathcal{A}\right)u < C \quad \text{and} \quad \left(-\frac{\partial}{\partial t} + \widehat{\mathcal{A}}\right)\widehat{u} < \widehat{C}$$

hold for certain positive constants C and $\widehat{C}$.

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