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VARIATIONAL DIRICHLET PROBLEM FOR ELLIPTIC OPERATORS IN THE WHOLE SPACE WITH UNCOORDINATED DEGENERACY OF COEFFICIENTS

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Abstract. In paper we investigate solvability of the variational Dirichlet problem for a class of elliptic differential operators in the whole space. The sesquilinear form associated with the operator under study is represented as a finite sum of auxiliary sesquilinear forms and the concept of ¡¡leading form¿¿ is introduced. Depending on the behavior of the coefficients of the leading forms, the main weighted space of differentiable functions of many real variables in the entire space is introduced. The solvability of the variational Dirichlet problem is studied in this space.

1. Introduction

The solvability of the variational Dirichlet problem for degenerate elliptic operators of higher order in the domain Ω of n-dimensional Euclidean space $\mathbb{R}^n$ is well studied in the case of bounded domain Ω and the coercivity of the corresponding sesquilinear form (see [1] - [8] and the bibliography therein). The case when the differential operators under study are generated using non-coercive sesquilinear forms is associated with many technical difficulties and was first considered in [9]. This case was later studied in [10] - [17]. The method developed in these papers is essentially based on the boundedness of the domain Ω in which the differential operator under study is specified. The improvement of this method in [16], [18] made it possible to study differential operators defined in unbounded domains that are very close to bounded domains (a limit-cylindrical domain with zero diameter at infinity).

In the authors' papers [19] - [21] we studied the solvability of the variational Dirichlet problem for elliptic operators in the whole space $\mathbb{R}^n$, the corresponding forms of which may not satisfy the coercivity condition and whose coefficients have coordinated power-law degeneracy at infinity. In this case, the main solution space is determined using the degree of degeneracy of the leading coefficients.

Here we consider in detail the case of uncoordinated degeneracy of the coefficients of the operator under study. In this case, we introduce the concept of "leading

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forms" and show that the main space of solutions to the Dirichlet problem is determined only by the degrees of degeneracy of the coefficients of leading forms. The coefficients of non-leading forms do not affect the definition of the main solution space.

2. Statement of the main result

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x = (x_1, x_2, \dots, x_n)$ and let $k = (k_1, k_2, \cdots, k_n)$ – multi-index and $|k| = k_1 + k_2 + \cdots + k_n$ – length of multi-index k. Let us denote by $u^{(k)}(x)$ the derivative of the function $u(x)$ of the multiindex k, generalized in the sense of S.L. Sobolev. Let r – a natural number and J – some subset of the set $\{0, 1, \dots, r\}$, with $r \in J$. Let $d(x) = (1 + |x|^2)^{-1/2}$ and α_j, $j \in J$, – real numbers. Consider the differential operator

$$L[u] = \sum_{|k|=|l|=j\in J} (-1)^j \left(d(x)^{2\alpha_j} a_{kl}(x) u^{(k)}(x)\right)^{(l)}, \tag{2.1}$$

which is understood in the sense of the theory of distributions on $\mathbb{R}^n$. It is assumed that the coefficients $a_{kl}(x)$, $x \in \mathbb{R}^n$, are bounded complex-valued functions.

Definition 1. The degeneracy of the coefficients of an operator (2.1) is said to be **coordinated** if there is a number α such that $\alpha_j = \alpha - j + r$ for all $j \in J$. Otherwise it is called **uncoordinated**.

This work is devoted to studying the solvability of the variational Dirichlet problem for a differential operator (2.1) in the case of uncoordinated degeneracy of its coefficients. The formulation of the variational Dirichlet problem for the operator (2.1) is associated with the following integro-differential sesquilinear form

$$B[u, v] = \sum_{|k|=|l|=j\in J} \int d^{2\alpha_j}(x) a_{kl}(x) u^{(k)}(x) \overline{v^{(l)}(x)} dx. \tag{2.2}$$

Here and below, all integrals are taken over the whole space $\mathbb{R}^n$ and it is assumed that the form (2.2), in the general case, may not satisfy the coercivity condition, which is understood in the sense of Definition 2.0.1 of the paper [7]: if H_0 – is a Hilbert space with scalar product $(\cdot, \cdot)_0$ and norm $\|\cdot\|_0$, H_+ – another Hilbert space with norm $\|\cdot\|_+$, densely embedded in H_0, then the sesquilinear form $P[u, v]$ defined in H_+ is called H_+-coercive with respect to H_0 if there are numbers $\mu_0 \in R$, $\delta_0 > 0$ such that

$$\operatorname{Re} P[u, u] + \mu_0 \|u\|_0^2 \geq \delta_0 \|u\|_+^2$$

for all $u \in H_+$.

We represent the form (2.2) as

$$B[u, v] = \sum_{j\in J} B_j[u, v], \text{ where } B_j[u, v] = \sum_{|k|=|l|=j} \int d^{2\alpha_j}(x) a_{kl}(x) u^{(k)}(x) \overline{v^{(l)}(x)} dx$$

and introduce the following definition:

Definition 2. The form $B_r[u, v]$ is called **leading form**. In what follows, for convenience, we denote r by j_1 and define other leading forms by induction. Let the leading forms $B_{j_1}[u, v], \dots, B_{j_{m-1}}[u, v]$ already be specified and let j_m be the greatest number of set J, less than j_{m-1}, for which the inequality $\alpha_{j_m} + j_m <$

$\min_{1\le h\le m-1}(\alpha_{j_h}+j_h)$. holds. Then the form $B_{j_m}[u,\, v]$ will also be called **leading form**. If there is no such j_m, then only the forms $B_{j_1}[u,\, v]$, $\ldots$, $B_{j_{m-1}}[u,\, v]$ are called **leading form**, and $m-1$ will be denoted by m.

Before formulating the variational Dirichlet problem for the operator (2.1), we define the main functional space in which the solution to this problem is sought. For $j\in J$ we denote by $V^j_{2,\alpha_j}(\mathbb{R}^n)$ the space of functions $u(x)$ defined in the whole space $\mathbb{R}^n$ which have all derivatives of the order j generalized in the sense of S.L. Sobolev with finite norm

$$\|u; V^j_{2;\,\alpha_j}(\mathbb{R}^n)\| = \left\{ \sum_{|k|=j} \int d^{2\alpha_j}(x)|u^{(k)}(x)|^2dx + \int d^{2(\alpha_j+j)}(x)|u(x)|^2dx \right\}^{1/2}.$$

Let $\delta = \min_{1\le h\le m}\{\alpha_{j_h} + j_h\}$ and $L_{2,\,\delta}(\mathbb{R}^n)$ be a Hilbert space with inner product

$$(u,\, v)_\delta = \int d^{2\delta}(x)u(x)\overline{v(x)}dx. \tag{2.3}$$

Let $\|f;\, L_{2,\,\delta}(\mathbb{R}^n)\|$ denote the norm of the space $L_{2,\,\delta}(\mathbb{R}^n)$ generated by the scalar product (2.3).

We introduce the space $\mathbb{H}_+$ of complex-valued functions $u(x)$, $x\in\mathbb{R}^n$, with finite norm

$$\|u;\, \mathbb{H}_+\| = \left\{ \sum_{h=0}^{m} \left\|u;\, V^{j_h}_{2;\alpha_{j_h}}(\mathbb{R}^n)\right\|^2 \right\}^{1/2}.$$

From the property of the space $V^j_{2,\,\alpha_j}(\mathbb{R}^n)$ (see, for example, [7]) it follows that the set $C_0^\infty(\mathbb{R}^n)$ is dense in the space $\mathbb{H}_+$ and $\|f;\, L_{2,\,\delta}(\mathbb{R}^n)\| \le \|u;\, \mathbb{H}_+\| \quad \forall u\in\mathbb{H}_+$.

The symbol $\mathbb{H}_-$ denotes the completion of the space $L_{2,\,\delta}(\mathbb{R}^n)$ according to the norm

$$\|f;\, \mathbb{H}_-\| = \sup \frac{|(f,\, u)_\delta|}{\|u;\, \mathbb{H}_+\|},$$

where the supremum is taken over all $u\in\mathbb{H}_+$ such that $\|u;\, \mathbb{H}_+\| = 1$. Elements from $\mathbb{H}_-$ are identified with the corresponding antilinear continuous functionals over $\mathbb{H}_+$. The action of the functional $F\in\mathbb{H}_-$ on the function $u\in\mathbb{H}_+$ will be denoted by the symbol $\langle F,\, v\rangle$.

Now let us formulate the variational Dirichlet problem for the operator (2.1).

Problem D_λ. For a given functional $F\in\mathbb{H}_-$ it is required to find a solution $u(x)$ to the equation

$$B[u,\, v] + \lambda(u,\, v)_\delta =< F,\, v > \quad \forall v\in C_0^\infty(\mathbb{R}^n),$$

which belongs to the space $\mathbb{H}_+$.

Before formulating the main result of our paper on the solvability of the problem D_λ for each $h=\overline{1,\, m}$ we introduce the function

$$A_h(x,\, \zeta) = \sum_{|k|=|l|=j_h} a_{kl}(x)\zeta_k\overline{\zeta_l},$$

where $x\in\mathbb{R}^n$ and $\zeta = \{\zeta_k\}_{|k|=j_h}$ – an a set of complex numbers.

Further we will assume that the function $\arg z$ takes its values on the interval $(-\pi,\, \pi]$.

Theorem. *Let for each $h \in \{1, \ldots, m\}$ there exist a number $\psi_h \in (o, \pi)$ and a nonzero complex-valued everywhere continuous function $\gamma_h(x)$, $x \in \mathbb{R}^n$, such that for all $x \in \mathbb{R}^n$, $\zeta = \{\zeta_k\}_{|k|\le r} \subset \mathbb{C}$ the following inequalities hold*

$$|arg\, A_h(x, \zeta)| < \psi_h, \tag{2.4}$$

$$\sum_{|k|=r} |\zeta_k|^2 \le M Re\, \{\gamma_h(x) A_h(x, \zeta)\}\,. \tag{2.5}$$

Here M is some positive number.

Let for any number $\nu > 0$ there exist $R_\nu > 0$ such that $|\gamma_h(x) - \gamma_h(y)| < \nu$ for any $h = \overline{1, m}$ and all $x, y \in \mathbb{R}^n$ such that $|x| > R_\nu$, $|y| > R_\nu$.

Then there is a number $\lambda_0 \ge 0$ such that if $\lambda \ge \lambda_0$, then for any given functional $F \in \mathbb{H}_-$ the problem D_λ has a unique solution and at the same time a following inequality holds

$$\|u; \mathbb{H}_+\| \le M_0 \|F; \mathbb{H}_-\|\,, \tag{2.6}$$

where the number $M_0 > 0$ does not depend on $\lambda \in [\lambda_0, +\infty)$ and on the functional F.

3. Proof of the main theorem

Without loss of generality, we can assume that the numbers φ_h and the functions $\gamma_h(x)$ under conditions (2.4), (2.5) do not depend on h. Therefore, without loss of generality, we will further assume that

$$\psi_1 = \psi_2 = \cdots = \psi_p = \psi, \quad \gamma_1(x) = \gamma_2(x) = \cdots = \gamma_p(x) = \gamma(x). \tag{3.1}$$

Lemma 1 (See [21, Lemma 3.1]). *Let the function $\gamma(x)$, $x \in \mathbb{R}^n$, be the same as in the previous section, and let ν be a sufficiently small positive number. Then there are non-negative functions $\varphi_m(x) \in C_0^\infty(\mathbb{R}^n)$, $\eta_m(x) \in C_0^\infty(\mathbb{R}^n)$, $m = 1, 2, \ldots$, such that:*

a) the system of functions $\left\{\varphi_m^2(x)\right\}_{m=1}^\infty$ forms a partition of the unit in the space $\mathbb{R}^n$ with finite multiplicity, that is

$$\sum_{m=1}^\infty \varphi_m^2(x) \equiv 1,\ x \in \mathbb{R}^n,$$

and if $\chi_m(x)$ is the characteristic function of the set $supp\,\varphi_m$, then there is a finite number Λ_n, depending only on n, such that

$$1 \le \sum_{m=1}^\infty \chi_m(x) \le \Lambda_n \quad \text{for all } x \in \mathbb{R}^n;$$

b) the function $\eta_m(x)$ vanishes to one in some neighborhood of the set $supp\,\varphi_m(x)$ and $0 \le \eta_m \le 1$ for all $x \in \mathbb{R}^n$;

c) the derivatives of the functions $\varphi_m(x)$, $\eta_m(x)$, $m = 1, 2, \ldots$, satisfy the following inequalities

$$\left|\varphi_m^{(k)}(x)\right| \le C_1\, d^{|k|}(x), \quad \left|\eta_m^{(k)}(x)\right| \le C_1\, d^{|k|}(x),\ |k| \le r,$$

positive numbers C_1, C_2 do not depend on m and r;

d) $|\gamma(x) - \gamma(y)| < \nu$ for all $x, y \in supp\,\eta_m$ $(m = 1, 2, \ldots)$.

Let ν be a sufficiently small positive number, and let $\varphi_m(x)$, $\eta_m(x)$ $(m = 1,\, 2\,3,\, \ldots)$ – the same non-negative functions as in Lemma 1. In each set $\operatorname{supp}\varphi_m$ $(m = 1,\, 2,\, 3,\, \ldots)$ we fix the point x_m and consider the sesquilinear form

$$B^{(0)}_{\lambda;m}[u,v] = \sum_{j\in J} B^{(0)}_{m,\,j}[u,v] + \lambda \int d^{2\delta}(x)u(x)\overline{v(x)}dx, \tag{3.2}$$

where

$$B^{(0)}_{m,\,j}[u,v] = \sum_{|k|=|l|=j} \int d^{2\alpha_j}(x)a^{(0)}_{klm}(x)u^{(k)}(x)\overline{v^{(l)}(x)}dx,$$

$$a^{(0)}_{klm}(x) = (1-\eta_m(x))\gamma(x_m)a_{kl}(x_m) + \eta_m(x)\gamma(x)a_{kl}(x) \tag{3.3}$$

for $|k| = |l| = j_h$, $h = \overline{1,\, p}$, and

$$a^{(0)}_{klm}(x) = (1-\eta_m(x))a_{kl}(x_m) + \eta_m(x)a_{kl}(x)$$

for $|k| = |l| = i_s$, $s = \overline{1,\, q}$.

Further, for convenience of writing, we present the form (3.2) as

$$B^{(0)}_{\lambda;m}[u,v] = \sum_{h=1}^{p} B^{(0)}_{m,\,j_h}[u,v] + \sum_{s=1}^{q} B^{(0)}_{m,\,i_s}[u,v]. \tag{3.4}$$

From the boundedness of the coefficients $a_{kl}(x)$, $|k|,|l| \le r$, it follows that the coefficients $a^{(0)}_{klm}(x)$ are bounded. Therefore, applying the Cauchy-Bunyakovsky inequality for leading forms we have

$$\left|B^{(0)}_{m,\,j_h}[u,v]\right| \le M_0\|u; V^{j_h}_{2;\alpha_{j_h}}(\mathbb{R}^n)\|\cdot\|v; V^{j_h}_{2;\alpha_{j_h}}(\mathbb{R}^n)\| \le M_0\,\|u;\,\mathbb{H}_+\|\,\|v;\,\mathbb{H}_+\|\,. \tag{3.5}$$

Now consider the form $B^{(0)}_{m,\,i_s}[u,v]$, which is not the leading form. According to the Definition 2, j_p is an index of the highest form if $j_p < j_{p-1}$ and $\alpha_{j_p} + j_p < \min_{1\le h\le p-1}(\alpha_{j_h} + j_h)$. Therefore, for any index of the non-leading form i_s there is an index of the leading form j_h such that

$$j_h > i_s, \quad \alpha_{i_s} + i_s \ge \alpha_{j_h} + j_h. \tag{3.6}$$

Taking this into account, by virtue of the embedding theorem for the spaces $V^r_{2;\,\alpha}(\mathbb{R}^n)$ (see, for example, [21, Theorem 2.1]), we have

$$\left\|u;\, V^{j_h}_{2,\,\alpha_{j_h}}(\mathbb{R}^n)\right\| \ge \left\|u;\, V^{i_s}_{2,\,\alpha_{j_h}+j_h-i_s}(\mathbb{R}^n)\right\|.$$

Since $0 < d(x) \le 1, \forall x \in \mathbb{R}^n$, then by inequality (3.6) we have

$$\left\|u;\, V^{i_s}_{2,\,\alpha_{j_h}+j_h-i_s}(\mathbb{R}^n)\right\| \ge \left\|u;\, V^{i_s}_{2,\,\alpha_{i_s}}(\mathbb{R}^n)\right\|.$$

Hence, by virtue of the definition of the space $\mathbb{H}_+$, it follows that

$$\left\|u;\, V^{i_s}_{2,\,\alpha_{i_s}}(\mathbb{R}^n)\right\| \le M_1\left\|u;\, V^{j_h}_{2,\,\alpha_{j_h}}(\mathbb{R}^n)\right\| \le M_1\,\|u;\,\mathbb{H}_+\|\,.$$

Therefore

$$\left|B^{(0)}_{m,\,i_s}[u,v]\right| \le M_2\|u; V^{i_s}_{2;\alpha_{i_s}}(\mathbb{R}^n)\|\cdot\|v; V^{i_s}_{2;\alpha_{i_s}}(\mathbb{R}^n)\| \le M_2\,\|u;\,\mathbb{H}_+\|\,\|v;\,\mathbb{H}_+\|\,. \tag{3.7}$$

Note also the obvious inequality

$$\left|\int d^{2\delta}(x)u(x)\overline{v(x)}dx\right| \leq \|u;\, \mathbb{H}_+\| \, \|v;\, \mathbb{H}_+\| \tag{3.8}$$

Now applying the inequalities (3.5), (3.7), (3.8), from (3.4) we have

$$\left|B^{(0)}_{\lambda;m}[u,v]\right| \leq (M_3 + |\lambda|)\, \|u;\, \mathbb{H}_+\| \, \|v;\, \mathbb{H}_+\| \tag{3.9}$$

for all $u, v \in \mathbb{H}_+$.

From condition (2.5) it follows that

$$\operatorname{Re}\left\{\gamma(x_m) \sum_{|k|=|l|=j_h} a_{kl}(x_m)\zeta_i\overline{\zeta_j}\right\} \geq c \sum_{|k|=j_h} |\zeta_k|^2,$$

$$\operatorname{Re}\left\{\gamma(x) \sum_{|k|=|l|=j_h} a_{kl}(x)\zeta_i\overline{\zeta_j}\right\} \geq c \sum_{|k|=j_h} |\zeta_k|^2$$

for all $m = 1,\, 2,\, 3,\, \ldots,\, x \in \mathbb{R}^n$ and for any set of complex numbers $\zeta = \{\zeta_k\}_{|k|\leq r} \subset \mathbb{C}$. Due to these inequalities, it follows from (3.3) that

$$\operatorname{Re}\left\{\sum_{|k|=|l|=j_h} a^{(0)}_{kl}(x)\zeta_i\overline{\zeta_j}\right\} \geq c \sum_{|k|=j_h} |\zeta_k|^2$$

for all $m = 1,\, 2,\, 3,\, \ldots,\, x \in \mathbb{R}^n$, $\zeta = \{\zeta_k\}_{|k|=j_h} \subset \mathbb{C}$.

Substituting $\zeta_k = d^{\alpha_{j_h}}(x)u^{(k)}(x)$ into this inequality after integrating over $\mathbb{R}^n$ we obtain

$$\operatorname{Re} B^{(0)}_{m,\, j_h}[u,u] \geq C_0 \sum_{|k|=j_h} \int d^{2\alpha_{j_h}}(x) \left|u^{(k)}(x)\right|^2 dx. \tag{3.10}$$

From this inequality and from the definition of the norm of the space $V^r_{2;\alpha}(\mathbb{R}^n)$ it follows that

$$\operatorname{Re} B^{(0)}_{m,\, j_h}[u,u] + \lambda_h \left\|u;\, L_{2,\, \alpha_{j_h}+j_h}(\mathbb{R}^n)\right\|^2 \geq C_0 \left\|u;\, V^{j_h}_{2,\, \alpha_{j_h}}(\mathbb{R}^n)\right\|^2 \tag{3.11}$$

for all $u \in C^\infty_0(\mathbb{R}^n)$.

Further, in the process of proving the main theorem, we will repeatedly use the inequality

$$\left\|u;\, L_{2,\, \alpha_j+j}(\mathbb{R}^n)\right\| \leq M_1 \left\|u;\, L_{2,\, \delta}(\mathbb{R}^n)\right\| \quad \forall j \in J,\; u \in C^\infty_0(\mathbb{R}^n),$$

which occurs due to the inequality $d^\delta(x) \geq d^{\alpha_j+j}(x),\; j \in J,\; x \in \mathbb{R}^n$.

Hence, due to the inequality (3.11) and the definition of the space $\mathbb{H}_+$, it follows that

$$\operatorname{Re} \sum_{h=1}^{p} B^{(0)}_{m,\, j_h}[u,u] + \lambda'_0 \left\|u;\, L_{2,\, \delta}(\mathbb{R}^n)\right\|^2 \geq c'_0 \left\|u;\, \mathbb{H}_+\right\|^2,\; u \in C^\infty_0(\mathbb{R}^n),$$

where λ'_0, c'_0 are some positive numbers.

Now consider the non-leading form $B^{(0)}_{m,\,i_s}[u,v]$. Due to the boundedness of the coefficients of this form, we have

$$\left|B^{(0)}_{m,\,i_s}[u,v]\right| \le M_2 \left\|u;\, L^{i_s}_{2;\,\alpha_{i_s}}(\mathbb{R}^n)\right\| \left\|v;\, L^{i_s}_{2;\,\alpha_{i_s}}(\mathbb{R}^n)\right\|. \tag{3.12}$$

Next, note that for any index of the non-leading form i_s there is an index of the leading form j_h such that the inequalities are satisfied $j_h > i_s, \quad \alpha_{i_s}+i_s \ge \alpha_{j_h}+j_h$.

On the other hand, from [23, Lemma 2.2], in particular, it follows that for any $\tau > 0$ and all $u \in C_0^\infty(\mathbb{R}^n)$ the following inequality is valid

$$\sum_{|k|=i_s} \left\|d^{-i_s+\alpha_{j_h}+j_h} D^k u;\, L_{2,}(\mathbb{R}^n)\right\| \le$$

$$\le \tau \|u; L^{j_h}_{2;\alpha_{j_h}}(\mathbb{R}^n)\| + c_0\tau^{-\mu} \left\|u;\, L_{2,\,\alpha_{j_h}+j_h}(\mathbb{R}^n)\right\|,$$

where $\mu = i_s/(j_h - i_s)$. From here, due to the inequality $\alpha_{i_s} \ge -i_s + \alpha_{j_h} + j_h$, it follows that

$$\left\|u;\, L^{i_s}_{2;\,\alpha_{i_s}}(\mathbb{R}^n)\right\| \le \tau \|u; L^{j_h}_{2;\alpha_{j_h}}(\mathbb{R}^n)\| + c_0\tau^{-\mu} \left\|u;\, L_{2,\,\alpha_{j_h}+j_h}(\mathbb{R}^n)\right\|, \tag{3.13}$$

where μ – finite positive number.

Applying inequality (3.13), from (3.12) we have

$$\left|B^{(0)}_{m,\,i_s}[u,u]\right| \le \tau^2 \|u; L^{j_h}_{2;\alpha_{j_h}}(\mathbb{R}^n)\|^2 + c_0\tau^{-2\mu} \left\|u;\, L_{2,\,\alpha_{j_h}+j_h}(\mathbb{R}^n)\right\|^2. \tag{3.14}$$

Next, using inequalities (3.10), (3.14), we obtain

$$\operatorname{Re} B^{(0)}_{m,\,j_h}[u,u] + \operatorname{Re} B^{(0)}_{m,\,i_s}[u,u] \ge \operatorname{Re} B^{(0)}_{m,\,j_h}[u,u] - \left|B^{(0)}_{m,\,i_s}[u,u]\right| \ge$$

$$\ge \left(C_0 - \tau^2\right) \|u; L^{j_h}_{2;\alpha_{j_h}}(\mathbb{R}^n)\|^2 - c_0\tau^{-2\mu} \left\|u;\, L_{2,\,\alpha_{j_h}+j_h}(\mathbb{R}^n)\right\|^2.$$

Then, choosing a suitable value for the parameter τ, we find

$$\operatorname{Re} B^{(0)}_{m,\,j_h}[u,u] + \operatorname{Re} B^{(0)}_{m,\,i_s}[u,u] + \lambda'_h \left\|u;\, L_{2,\,\alpha_{j_h}+j_h}(\mathbb{R}^n)\right\|^2 \ge C_0 \left\|u;\, V^{j_h}_{2,\,\alpha_{j_h}}(\mathbb{R}^n)\right\|^2.$$

Further, taking into account the definitions of the forms $B^{(0)}_{m,\,j_h}[u,u],\ B^{(0)}_{m,\,i_s}[u,u]$ and the space $\mathbb{H}_+$, we obtain the inequality

$$\operatorname{Re} B^{(0)}_{\lambda;m}[u,u] \ge c''_0 \|u;\, \mathbb{H}_+\|^2, \ (\lambda \ge \lambda''_0,\ m = 1,\, 2,\, 3,\, \ldots, \quad u \in \mathbb{H}_+), \tag{3.15}$$

where $\lambda''_0,\ c''_0$ are some positive constants.

Now consider the sesquilinear form

$$\mathcal{B}^{(0)}_{\lambda;m}[u,v] = \sum_{h=1}^{p} \sum_{|k|=|l|=j_h} \int d^{2\alpha_j}(x)\widehat{a}_{klm}(x)u^{(k)}(x)\overline{v^{(l)}(x)}dx + \sum_{s=1}^{q} B^{(0)}_{m,\,i_s}[u,v] +$$

$$+ \lambda \int d^{2\delta}(x)u(x)\overline{v(x)}dx,$$

where $\widehat{a}_{klm}(x) = [(1-\eta_m(x))a_{kl}(x_m) + \eta_m(x)a_{kl}(x)]\gamma_h(x_m)$ for $|k| = |l| = j_h,\ h = \overline{1,\, p}$. Since $a^{(0)}_{klm}(x) - \widehat{a}_{klm}(x) = \eta_m(x)(\gamma_h(x) - \gamma_h(x_m))a_{kl}(x)$ $(|k| = |l| = j_h,\ h =$

$\overline{1,\,p}$) and coefficients $a_{kl}(x)$ are bounded, then using the Cauchy-Bunyakovsky inequality it is proved that

$$|B^{(0)}_{\lambda;m}[u,v] - \mathcal{B}^{(0)}_{\lambda;m}[u,v]| \le M\,\Lambda\,\|u;\,\mathbb{H}_+\|\cdot\|v;\,\mathbb{H}_+\|$$

for all $u,v\in C_0^\infty(\mathbb{R}^n)$. Here $\Lambda=\sup|\eta_m(x)(\gamma_m(x)-\gamma(x_m))|$, where the supremum is taken over all $x\in R_n$, $h=\overline{1,\,p}$ and all $m=1,\,2,\,3,\,\ldots$.

Applying this inequality, from (3.15) we find

$$C_0\|u;\,\mathbb{H}_+\|^2 \le \operatorname{Re}\left(B^{(0)}_{\lambda;m}[u,u] - \mathcal{B}^{(0)}_{\lambda;m}[u,u]\right) + \operatorname{Re}\mathcal{B}^{(0)}_{\lambda;m}[u,u] \le$$

$$\le \operatorname{Re}\mathcal{B}^{(0)}_{\lambda;m}[u,u] + M\,\Lambda\,\|u;\,\mathbb{H}_+\|^2.$$

Since $|\eta_m(x)(\gamma_h(x)-\gamma_h(x_m))|<\nu,\quad h=\overline{1,\,p},\quad m=1,\,2,\ldots$ and ν is a sufficiently small positive number, then from the resulting inequality it follows that

$$c_0\|u;\,\mathbb{H}_+\|^2 \le \operatorname{Re}\mathcal{B}^{(0)}_{\lambda;m}[u,u] \tag{3.16}$$

for all $u\in C_0^\infty(\mathbb{R}^n)$. Here $\lambda\ge\lambda_0''$ and λ_0'' are the same positive number as in (3.15).

Enter the following sesquilinear form

$$\mathcal{B}_{\lambda;m}[u,v] = \sum_{h=1}^{p}\sum_{|k|=|l|=j_h}\int d^{2\alpha_j}(x)a_{klm}(x)u^{(k)}(x)\overline{v^{(l)}(x)}dx+$$

$$+\gamma^{-1}(x_m)\sum_{s=1}^{q}B^{(0)}_{m,\,i_s}[u,v] + \lambda\int d^{2\delta}(x)u(x)\overline{v(x)}dx, \tag{3.17}$$

where $a_{klm}(x)=(1-\eta_m(x))a_{kl}(x_m)+\eta_m(x)a_{kl}(x),\ |k|=|l|=j_h,\ h=\overline{1,\,p}$.

We note that $\mathcal{B}^{(0)}_{\lambda;m}[u,v]=\gamma(x_m)\mathcal{B}_{\lambda_m;m}[u,v]$, where $\lambda_m=\lambda\gamma^{-1}(x_m)$. Therefore, from inequality (3.16) it follows that for $\lambda\ge\lambda_0$, where λ_0 is some sufficiently large number, the inequality

$$c_0\|u;\,\mathbb{H}_+\|^2 \le \operatorname{Re}\{\gamma(x_m)\mathcal{B}_{\lambda;m}[u,u]\}\,,\ u\in C_0^\infty(\mathbb{R}^n) \tag{3.18}$$

holds.

Further, without loss of generality, we will assume that the number (see (3.1)) $\psi_h=\psi$ in condition (2.4) is such that $\psi>\pi/2$. By virtue of (2.4), inequality (2.5) will also hold if $\gamma(x)$ is replaced by $\exp(i\theta(x))$, where

$$\theta(x)=\min\{\psi-\pi/2,\,|\arg\gamma(x)|\}\,(\operatorname{sign}\arg\gamma(x)).$$

Therefore, from inequality (3.18) it follows that

$$c_0\|u;\,\mathbb{H}_+\|^2 \le \operatorname{Re}\{\exp(i\theta_m)\mathcal{B}_{\lambda;m}[u,u]\}\,,\ u\in C_0^\infty(\mathbb{R}^n). \tag{3.19}$$

Here and below $\theta_m=\theta(x_m),\qquad m=1,\,2,\ldots$.

Doing the same thing as in the proof of inequality (3.9), we find

$$|\mathcal{B}_{\lambda;m}[u,v]| \le (M_0+|\lambda|)\|u;\,\mathbb{H}_+\|\cdot\|v;\,\mathbb{H}_+\|,\quad u\in C_0^\infty(\mathbb{R}^n). \tag{3.20}$$

Inequalities (3.19), (3.20) allow us to apply the generalized Lax-Milgram theorem [7, theorem 2.0.1]). According to this theorem, there is an operator $\widetilde{\mathcal{R}}_m(\lambda)$ that implements a homeomorphism of the spaces $\mathbb{H}_+$ and $\mathbb{H}_-$ such that

$$\left\langle \widetilde{\mathcal{R}}_m(\lambda)u,\, v \right\rangle = \exp(i\theta_m)\mathcal{B}_{\lambda;m}[u, v], \quad v \in \mathbb{H}_+. \tag{3.21}$$

Let's denote $\mathcal{R}_m(\lambda) = \widetilde{\mathcal{R}}_m^{-1}(\lambda) : \mathbb{H}_- \to \mathbb{H}_+$. Then it follows from equality (3.21) that

$$\exp(i\theta_m)\mathcal{B}_{\lambda;m}[\mathcal{R}_m(\lambda)F, v] = \langle F, v\rangle, \quad F \in \mathbb{H}_-, v \in \mathbb{H}_+. \tag{3.22}$$

The operator $\mathcal{R}_m(\lambda)$ is bounded and

$$\|\mathcal{R}_m(\lambda)F; \mathbb{H}_+\| \le M_1 \|F; \mathbb{H}_-\|, \quad F \in \mathbb{H}_-,\, \lambda \ge \lambda_0. \tag{3.23}$$

Here the number M_1 does not depend on F and λ, and the number λ_0 is the same as in (3.18).

We introduce the operator

$$\mathcal{R}(\lambda) = \sum_{m=1}^{\infty} \exp(i\theta_m)\Phi_m \mathcal{R}_m(\lambda)\Phi_m, \tag{3.24}$$

which acts from $\mathbb{H}_-$ to $\mathbb{H}_+$. Here and below, the symbol Φ_m denotes the operator of multiplication by the function $\varphi_m(x)$.

Since the coefficients $a_{kl}(x)$ $(|k|, |l| \le r)$ are bounded, then by applying the Cauchy-Bunyakovsky inequality it is proved that

$$\left| B[u,\, v] + \lambda \int d^{2\delta}(x)u(x)\overline{v(x)}dx \right| \le (M_0 + \lambda)\|u; \mathbb{H}_+\| \cdot \|v; \mathbb{H}_+\|, \quad u,\, v \in \mathbb{H}_+.$$

Therefore, the operator $\mathbb{R}(\lambda)$ defined by the equality

$$\langle \mathbb{R}(\lambda)F, v\rangle = B[\mathcal{R}(\lambda)F, v] + \lambda \int d^{2\delta}(x) \left(\mathcal{R}(\lambda)F\right)(x)\overline{v(x)}dx \quad (\forall v \in \mathbb{H}_+) \tag{3.25}$$

acts from $\mathbb{H}_-$ to $\mathbb{H}_-$.

According to our constructions, the functions $\varphi_m^2(x)$, $m = 1, 2, 3, \ldots$, form a partition of the unit $\mathbb{R}^n$, therefore for all $F \in L_{2,\,\delta}(\mathbb{R}^n)$ and all $v \in \mathbb{H}_+$ the following equalities hold

$$\langle F,\, v\rangle = (F,\, v)_\delta = \sum_{m=1}^{\infty} (\varphi_m F,\, \varphi_m v)_\delta. \tag{3.26}$$

Since $a_{klm}(x) = (1-\eta_m(x))a_{kl}(x_m)+\eta_m(x)a_{kl}(x)$, and the function $\eta_m(x)$ identity equals to one in some neighborhood of the set $\operatorname{supp}\varphi_m$, then the functions $a_{klm}(x)$ and $a_{kl}(x)$ on the set supp , φ_m match. Therefore, from equalities (3.24) and (3.25) it follows that

$$\langle \mathbb{R}(\lambda)F, v\rangle =$$

$$= \sum_{m=1}^{\infty} \exp(i\theta_m) \left\{ \sum_{|k|=|l|=j\in J} \int d^{2\alpha_j}(x)a_{klm}(x)D^k(\varphi_m \mathcal{R}_m(\lambda)\Phi_m F)(x)\overline{v^{(l)}(x)}dx + \right.$$

$$\left. + \lambda \int d^{2\delta}(x)(\mathcal{R}_m(\lambda)\Phi_m F)(x)\overline{\varphi_m(x)v(x)}dx \right\}. \tag{3.27}$$

Let $F \in L_{2,\delta}(\mathbb{R}^n)$. In equality (3.22) we replace F with $\varphi_m F$, and v with $\varphi_m v$:

$$\exp(i\theta_m)\mathcal{B}_{\lambda;m}[\mathcal{R}_m(\lambda)\Phi_m F, \varphi_m v] = (\varphi_m F, \varphi_m v)_\delta.$$

From here, taking into account the equality (3), it follows that

$$(\varphi_m F, \varphi_m v)_\delta = \exp(i\theta_m)\times$$
$$\times\left\{\sum_{|k|=|l|=j\in J}\int d^{2\alpha_j}(x)a_{klm}(x)D^k\left((\mathcal{R}_m(\lambda)\Phi_m F)(x)\right)\overline{D^l\left(\varphi_m(x)v(x)\right)}dx+\right.$$
$$\left.+\lambda\int d^{2\delta}(x)(\mathcal{R}_m(\lambda)\Phi_m F)(x)\overline{\varphi_m(x)v(x)}dx\right\}.$$

Summing this equality over m from 1 to infinity, by virtue of (3.26), we have

$$\langle F,\, v\rangle = (F,\, v)_\delta = \sum_{m=1}^{\infty}\exp(i\theta_m)\times$$
$$\times\left\{\sum_{|k|=|l|=j\in J}\int d^{2\alpha_j}(x)a_{klm}(x)D^k\left((\mathcal{R}_m(\lambda)\Phi_m F)(x)\right)\overline{D^l\left(\varphi_m(x)v(x)\right)}dx+\right.$$
$$\left.+\lambda\int d^{2\delta}(x)(\mathcal{R}_m(\lambda)\Phi_m F)(x)\overline{\varphi_m(x)v(x)}dx\right\}.$$

From here and from (3.27) it follows that

$$\langle\mathbb{R}(\lambda)F, v\rangle - \langle F,\, v\rangle = \sum_{m=1}^{\infty}\exp(i\theta_m)\left\{\sum_{|k|=|l|=j\in J}\int d^{2\alpha_j}(x)a_{klm}(x)\times\right.$$
$$\times\left\{D^k\left((\varphi_m\mathcal{R}_m(\lambda)\Phi_m F)(x)\right)\overline{v^{(l)}(x)}-\right.$$
$$\left.\left.-D^k\left((\mathcal{R}_m(\lambda)\Phi_m F)(x)\right)\overline{D^l\left(\varphi_m(x)v(x)\right)}\right\}dx\right\}. \quad (3.28)$$

We denote

$$U_{m,\lambda}(x) = (\mathcal{R}_m(\lambda)\Phi_m F)(x), \quad m = 1,\, 2,\, \ldots. \quad (3.29)$$

Then, by virtue of equality (3.28), we have

$$\langle\mathbb{R}(\lambda)F, v\rangle - \langle F,\, v\rangle = \mathbb{K}_\lambda[F,\, v] + \mathbb{L}_\lambda[F,\, v], \quad (3.30)$$

where

$$\mathbb{K}_\lambda[F,\, v] = \sum_{m=1}^{\infty}\sum_{|k|=|l|=j\in J}\exp(i\theta_m)\times$$
$$\times{\sum_k}^{(1)} C_{k'}^{k''}\int d^{2\alpha_j}(x)a_{klm}(x)\varphi_m^{(k')}(x)U_{m,\lambda}^{(k'')}(x)\overline{v^{(l)}(x)}dx, \quad (3.31)$$

$$\mathbb{L}_\lambda[F,\,v] = \sum_{m=1}^{\infty} \sum_{|k|=|l|=j\in J} \exp(i\theta_m)\times$$
$$\times \sum\nolimits_l^{(2)} C_{l'}^{l''} \int d^{2\alpha_j}(x) a_{klm}(x) U_{m,\lambda}^{(k)}(x) \overline{\varphi_m^{(l')}(x) v^{(l'')}(x)} dx. \quad (3.32)$$

Here the symbol $\sum_k^{(1)}$ denotes summation over multi-indices k', k'' such that $k = k' + k''$, $k' \neq 0$, and the symbol $\sum_l^{(2)}$ denotes summation over multi-indices l', l'' such that $l = l' + l''$, $l' \neq 0$.

Consider a symmetrical sesquilinear form

$$\widetilde{\mathcal{B}}_{\lambda;m}[u,v] = \frac{1}{2}\left\{\exp(i\theta_m)\mathcal{B}_{\lambda;m}[u,v] + \exp(-i\theta_m)\overline{\mathcal{B}_{\lambda;m}[v,u]}\right\}, \quad (3.33)$$
$$D(\widetilde{\mathcal{B}}_{\lambda;m}) = \mathbb{H}_+.$$

There is a self-adjoint operator $B_{\lambda;m}$ in the space $L_{2,\,\delta}(\mathbb{R}^n)$ generated by a symmetric form (3.33) such that

$$\left(B_{\lambda;m}^{1/2}u, B_{\lambda;m}^{1/2}v\right)_{\alpha+r} = \widetilde{\mathcal{B}}_{\lambda;m}[u,v], \quad u,v \in \mathbb{H}_+. \quad (3.34)$$

Lemma 2 *A) There is a non-negative number λ_0 such that for $\lambda \geq \lambda_0$ and all $j \in J$ for any multi-index k such that $|k| = j$, and any $m = 1, 2, 3, \ldots$ operator $d^{-|k|}D^k B_{\lambda;m}^{-1/2}$ is a bounded operator acting from $L_{2,\,\delta}(\mathbb{R}^n)$ to $L_{2,\,\alpha_j+j}(\mathbb{R}^n)$.*

B) if $|k| = j \in J$, $k = k' + k''$ and $|k'| \neq 0$, then there is a positive function $q(\lambda)$ such that $q(\lambda) \to 0$ for $\lambda \to \infty$ and

$$\left\| d^{-|k|}\varphi_m^{(k')} D^{k''} u;\, L_{2,\,\alpha_j+j}(\mathbb{R}^n) \right\| \leq q(\lambda) \|B_{\lambda;m}^{1/2}u; L_{2,\,\delta}(\mathbb{R}^n)\| \quad (3.35)$$

for all $u \in \mathbb{H}_+$

Proof. It follows from (3.34) that

$$\|B_{\lambda;m}^{1/2}u; L_{2,\,\delta}(\mathbb{R}^n)\|^2 = \mathrm{Re}\left\{\exp(i\theta_m)\mathcal{B}_{\lambda;m}[u,u]\right\} \quad (3.36)$$

Therefore, by virtue of the inequality (3.19) we have

$$\|B_{\lambda;m}^{1/2}u; L_{2,\,\delta}(\mathbb{R}^n)\| \geq c_0\|u; \mathbb{H}_+\| \qquad (\lambda \geq \lambda_0) \quad (3.37)$$

for all $u \in \mathbb{H}_+$. Hence, by virtue of the definition of the space $\mathbb{H}_+$, it follows that

$$\sum_{h=1}^{p} \left\| u;\, V_{2;\,\alpha_{j_h}}^{j_h}(\mathbb{R}^n) \right\| \leq M_0 \|B_{\lambda;m}^{1/2}u;\, L_{2,\,\delta}(\mathbb{R}^n)\| \quad (3.38)$$
$$(|k| = j \in J, \quad m = 1, 2, \ldots, \; u \in \mathbb{H}_+).$$

Since for any index i_s of a non-leading form there is an index j_h of a leading form such that

$$\left\| u;\, V_{2;\,\alpha_{i_s}}^{i_s}(\mathbb{R}^n) \right\| \leq M_1 \left\| u;\, V_{2;\,\alpha_{j_h}}^{j_h}(\mathbb{R}^n) \right\|, \; u \in C_0^\infty(\mathbb{R}^n),$$

than it follows from (3.38) that

$$\sum_{j\in J}^{p} \left\| u;\, V_{2;\,\alpha_j}^{j}(\mathbb{R}^n) \right\| \leq M_0 \|B_{\lambda;m}^{1/2}u;\, L_{2,\,\delta}(\mathbb{R}^n)\|.$$

Future it follows that

$$\left\| d^{-|k|} D^k u;\, L_{2,\,\alpha_j+j}(\mathbb{R}^n) \right\| \le M_0 \| B^{1/2}_{\lambda;m} u;\, L_{2,\,\delta}(\mathbb{R}^n)\|,$$

$$(|k| = j \in J, \quad m = 1,\, 2,\, \ldots,\; u \in \mathbb{H}_+).$$

Therefore, the operator $d^{-|k|} D^k B^{-1/2}_{\lambda;m}$ is a bounded operator acting from $L_{2,\,\delta}(\mathbb{R}^n)$ to $L_{2,\,\alpha_j+j}(\mathbb{R}^n)$.

We proceed to the proof of point B) of Lemma 2. Consider the case $0 \ne |k''| < |k| \le r$. Applying Lemma 1 we get

$$\left\| d^{-|k|}(x) \varphi_m^{(k')}(x) D^{k''} u(x);\, L_{2,\,\alpha_j+j}(\mathbb{R}^n) \right\| \le \\ \le \left\| d^{-|k''|} D^{k''} u;\, L_{2,\,\alpha_j+j}(\mathbb{R}^n) \right\|. \quad (3.39)$$

On the other hand, from [23, Lemma 2.2], in particular, it follows that for any $\tau > 0$ and all $u \in C_0^\infty(\mathbb{R}^n)$ the following inequality is valid

$$\left\| d^{-|k''|} D^{k''} u;\, L_{2,\,\alpha_j+j}(\mathbb{R}^n) \right\| \le \tau \| u; L^j_{2;\alpha}(\mathbb{R}^n)\| + c_0 \tau^{-\mu} \left\| u;\, L_{2,\,\alpha_j+j}(\mathbb{R}^n) \right\|,$$

where

$$\| u; L^j_{2,\,\alpha_j}(\mathbb{R}^n)\| = \left\{ \sum_{|l|=j} \int \left(d^{\alpha_j}(x) |u^{(l)}(x)| \right)^2 dx \right\}^{1/2}$$

and

$$\mu = |k''| / (j - |k''|). \quad (3.40)$$

Since $0 \ne |k''| < |k| = j$, then μ is a finite positive number.

From the above inequality and from (3.39) it follows that

$$\left\| d^{-|k|}(x) \varphi_m^{(k')}(x) D^{k''} u(x);\, L_{2,\,\alpha_j+j}(\mathbb{R}^n) \right\| \le \tau \| u; L^j_{2,\,\alpha_j}(\mathbb{R}^n)\| + \\ + c_0 \tau^{-\mu} \left\| u;\, L_{2,\,\alpha_j+j}(\mathbb{R}^n) \right\|.$$

By squaring this inequality, we obtain (we again denote $\sqrt{2}\tau$ by τ)

$$\left\| d^{-|k|}(x) \varphi_m^{(k')}(x) D^{k''} u(x);\, L_{2,\,\alpha_j+j}(\mathbb{R}^n) \right\|^2 \le \\ \le \tau^2 \| u; L^j_{2,\,\alpha_j}(\mathbb{R}^n)\|^2 + c_1 \tau^{-2\mu} \left\| u;\, L_{2,\,\alpha_j+j}(\mathbb{R}^n) \right\|^2.$$

Since $\delta = \min\limits_{1 \le h \le p} (\alpha_{j_h} + j_h)$ and for any index i_s of a non-leading form there is an index j_h of a leading form such that $\alpha_{i_s} + i_s \ge \alpha_{j_h} + j_h$, then $\delta = \min\limits_{j \in J} (\alpha_j + j)$. Therefore, the following inequality holds

$$\left\| u;\, L_{2,\,\alpha_j+j}(\mathbb{R}^n) \right\| \le \| u;\, L_{2,\,\delta}(\mathbb{R}^n)\|, \quad j \in J. \quad (3.41)$$

Next, applying the inequalities (3.37), (3.41), we have

$$\left\| d^{-|k|}(x) \varphi_m^{(k')}(x) D^{k''} u(x);\, L_{2,\,\alpha_j+j}(\mathbb{R}^n) \right\|^2 \le \\ \le \tau^2 \| B^{1/2}_{\lambda;m} u; L_{2,\,\delta}(\mathbb{R}^n)\|^2 + c_1 \tau^{-2\mu} \| u;\, L_{2,\,\delta}(\mathbb{R}^n)\|^2.$$

From here, by virtue of equality (3.36), it follows that

$$\left\| d^{-|k|}(x)\varphi_m^{(k')}(x)D^{k''}u(x);\, L_{2,\,\alpha_j+j}(\mathbb{R}^n)\right\|^2 \leq \\ \leq \tau^2 \mathrm{Re}\left\{\exp(i\theta_m)\mathcal{B}_{\lambda;m}[u,u]\right\} + c_1\tau^{-2\mu}\left\|u;\, L_{2,\,\delta}(\mathbb{R}^n)\right\|^2. \quad (3.42)$$

Using equality (3), we estimate the right-hand side of this inequality

$$\tau^2 \mathrm{Re}\left\{\exp(i\theta_m)\mathcal{B}_{\lambda;m}[u,u]\right\} + c_1\tau^{-2\mu}\left\|u;\, L_{2,\,\delta}(\mathbb{R}^n)\right\|^2 = \\ = \tau^2 \mathrm{Re}\left\{\exp(i\theta_m)\left(\sum_{|k|=|l|=j\in J}\int d^{2\alpha_j}(x)a_{klm}(x)u^{(k)}(x)\overline{u^{(l)}(x)}dx + \right.\right. \\ \left.\left. + \lambda\int d^{2\delta}(x)|u(x)|^2dx\right)\right\} + c_1\tau^{-2\mu}\int d^{2\delta}(x)|u(x)|^2dx \leq \\ \leq \tau^2 \mathrm{Re}\left\{\exp(i\theta_m)\left(\sum_{|k|=|l|=j\in J}\int d^{2\alpha_j}(x)a_{klm}(x)u^{(k)}(x)\overline{u^{(l)}(x)}dx + \right.\right. \\ \left.\left. + \Lambda(\lambda,\,\tau)\int d^{2\delta}(x)|u(x)|^2dx\right)\right\},$$

where $\Lambda(\lambda,\,\tau)$ is a continuous function satisfying the condition $\lambda + c_1\tau^{-2\mu-2} \leq \Lambda(\lambda,\,\tau)$. From (3.40) it follows that $\mu+1 = j/(j-|k''|)$. Since $0 \neq |k''| < j$, then $\Lambda(\lambda,\,\tau) \to \infty$ if $\tau \to 0$ or $\lambda \to \infty$. Therefore, from the inequality obtained above for $\lambda = 1/\tau$ and from (3.42) it follows that

$$\left\| d^{-|k|}(x)\varphi_m^{(k')}(x)D^{k''}u(x);\, L_{2,\,\alpha_j+j}(\mathbb{R}^n)\right\|^2 \leq \\ \leq \tau^2 \mathrm{Re}\left\{\exp(i\theta_m)\left(\sum_{|k|=|l|=j\in J}\int d^{2\alpha_j}(x)a_{klm}(x)u^{(k)}(x)\overline{u^{(l)}(x)}dx + \right.\right. \\ \left.\left. + p(\tau)\int d^{2\delta}(x)|u(x)|^2dx\right)\right\}, \quad (3.43)$$

where $p(\tau) = \Lambda(1/\tau,\,\tau)$. Note that $p(\tau) \to \infty$ for $\tau \to 0$. Let $q(\cdot)$ denote the inverse function of $p(\tau)$. Then for $\tau = q(\lambda)$, that is, for $\lambda = p(\tau)$, it follows from (3.43) that

$$\left\| d^{-|k|}(x)\varphi_m^{(k')}(x)D^{k''}u(x);\, L_{2,\,\alpha_j+j}(\mathbb{R}^n)\right\|^2 \leq \\ \leq q(\lambda)^2 \mathrm{Re}\left\{\exp(i\theta_m)\left(\sum_{|k|=|l|=j\in J}\int d^{2\alpha_j}(x)a_{klm}(x)u^{(k)}(x)\overline{u^{(l)}(x)}dx + \right.\right. \\ \left.\left. + \lambda\int d^{2\delta}(x)|u(x)|^2dx\right)\right\}.$$

Here a positive continuous function $q(\lambda)$ is defined for positive values of λ and such that $q(\lambda) \to 0$ for $\lambda \to \infty$. From the inequality obtained above, by virtue of equalities (3), (3.36), it follows (3.35). Statement B) of Lemma 2 for $|k''| \neq 0$ is proven.

Consider the case $|k''| = 0$. The inequality (3.35) in this case takes the form

$$\left\| d^{-|k|}\varphi_m^{(k)}(x)u(x);\, L_{2,\,\alpha_j+j}(\mathbb{R}^n)\right\| \leq q(\lambda)\|B_{\lambda;m}^{1/2}u; L_{2,\,\delta}(\mathbb{R}^n)\| \tag{3.44}$$

for all $u \in \mathbb{H}_+$; $|k| = j \in J$ and $q(\lambda) \to 0$ for $\lambda \to \infty$. By Lemma 1 the inequality (3.44) follows from the inequality

$$\left\|u;\, L_{2,\,\alpha_j+j}(\mathbb{R}^n)\right\| \leq q(\lambda)\|B_{\lambda;m}^{1/2}u; L_{2,\,\delta}(\mathbb{R}^n)\|, \tag{3.45}$$

that is proven below.

Let $\lambda > \lambda_0$, where λ_0 is the same positive number as in (3.37). Then using equality (3.33), we have

$$\|B_{\lambda;m}^{1/2}u; L_{2,\,\delta}(\mathbb{R}^n)\|^2 = \mathrm{Re}\,\{\exp(i\theta_m)\mathcal{B}_{\lambda;m}[u,u]\} =$$

$$= \mathrm{Re}\left\{\exp(i\theta_m)\left(\sum_{|k|=|l|=j\in J}\int d^{2\alpha_j}(x)a_{klm}(x)u^{(k)}(x)\overline{u^{(l)}(x)}dx+\right.\right.$$

$$\left.\left.+\lambda\int d^{2\delta}(x)|u(x)|^2dx\right)\right\} =$$

$$= \|B_{\lambda_0;m}^{1/2}u; L_{2,\,\delta}(\mathbb{R}^n)\|^2 + (\lambda-\lambda_0)\cos\theta_m\int d^{2\delta}(x)|u(x)|^2dx \geq$$

$$\geq (\lambda-\lambda_0)\cos\theta_m\int d^{2\delta}(x)|u(x)|^2dx.$$

It follows from here that

$$\|u;\, L_{2,\,\delta}(\mathbb{R}^n)\|^2 = \int d^{2\delta}(x)|u(x)|^2dx \leq \frac{1}{(\lambda-\lambda_0)\cos\theta_m}\|B_{\lambda;m}^{1/2}u; L_{2,\,\delta}(\mathbb{R}^n)\|^2.$$

Introducing the notation $q(\lambda) = 1/\sqrt{(\lambda-\lambda_0)\cos\theta_m}$, from the last inequality we obtain

$$\|u;\, L_{2,\,\delta}(\mathbb{R}^n)\| \leq q(\lambda)\|B_{\lambda;m}^{1/2}u; L_{2,\,\delta}(\mathbb{R}^n)\|.$$

From here, by virtue of the inequality (3.41), the inequality (3.45) follows.

□

Bilinear form $\exp(i\theta_m)\mathcal{B}_{\lambda;m}[u,v]$ satisfies the inequalities:

$$c_0\|u;\mathbb{H}_+\|^2 \leq \mathrm{Re}\,\{\exp(i\theta_m)\mathcal{B}_{\lambda;m}[u,u]\}\,, \quad u \in \mathbb{H}_+; \tag{3.46}$$

$$|\mathcal{B}_{\lambda;m}[u,v]| \leq (M_0 + |\lambda|)\|u;\mathbb{H}_+\|\cdot\|v;\mathbb{H}_+\|, \quad u,\, v \in \mathbb{H}_+. \tag{3.47}$$

The numbers c_0, $M_0 > 0$ in these inequalities do not depend on $u(x)$, $v(x)$. It follows from here that $D\left(\mathcal{B}_{\lambda;m}\right) = \mathbb{H}_+$.

According to inequalities (3.46), (3.47), the bilinear form $\exp(i\theta_m)\mathcal{B}_{\lambda;m}[u,v]$ is closed and sectorial. Therefore, using [24, ch. 6, Theorem 2.1], we obtain:

I) there is such an m – sectorial operator $A_{\lambda;m}$, which for all $u \in D(A_{\lambda;m}) \subset D(\mathcal{B}_{\lambda;m}) = \mathbb{H}_+$ and all $v \in \mathbb{H}_+$ the equality

$$\exp(i\theta_m)\mathcal{B}_{\lambda;m}[u,v] = (A_{\lambda;m}u,v)_\delta,$$

holds, where $(\cdot,\,\cdot)_\delta$ – the scalar product in space the $L_{2,\,\delta}(\mathbb{R}^n)$;

II) if $u \in \mathbb{H}_+$, $w \in L_{2,\,\delta}(\mathbb{R}^n)$ and $\exp(i\theta_m)\mathcal{B}_{\lambda;m}[u,v] = (w,\,v)_\delta$ for all v belonging to the kernel of the form $\mathcal{B}_{\lambda;m}$, then $u \in D(A_{\lambda;m})$ and $A_{\lambda;m}u = w$.

Let $f \in L_{2,\,\delta}(\mathbb{R}^n)$. Then $f \in \mathbb{H}_-$ and therefore $\mathcal{R}_m(\lambda)f \in \mathbb{H}_+$ and due to equality (3.22)

$$\exp(i\theta_m)\mathcal{B}_{\lambda;m}[\mathcal{R}_m(\lambda)f,v] = \langle f,v\rangle, \quad v \in \mathbb{H}_+. \tag{3.48}$$

By virtue of statement I), it follows from equality (3.48) that $A_{\lambda;m}\mathcal{R}_m(\lambda)f = f,\ \ \forall f \in L_{2,\,\delta}(\mathbb{R}^n)$. Hence,

$$\mathcal{R}_m(\lambda)f = A^{-1}_{\lambda;m}f, \quad f \in L_{2,\,\delta}(\mathbb{R}^n). \tag{3.49}$$

Let $B_{\lambda;m}$ is a self-adjoint operator in the space $L_{2,\,\delta}(\mathbb{R}^n)$ generated by a symmetric form (3.33). For all $u,v \in \mathbb{H}_+$ the equality $\left(B^{1/2}_{\lambda;m}u, B^{1/2}_{\lambda;m}v\right)_\delta = \widetilde{\mathcal{B}}_{\lambda;m}[u,v]$ holds. From here, with $u(x) = v(x)$, taking into account the equality (3.33), we obtain

$$\left\|B^{1/2}_{\lambda;m}u; L_{2,\,\delta}(\mathbb{R}^n)\right\|^2 = \operatorname{Re}\bigl\{\exp(i\theta_m)\mathcal{B}_{\lambda;m}[u,u]\bigr\} \ \ (\lambda \geq \lambda_0 > 0).$$

Next, applying inequality (3.19), we find

$$\left\|B^{1/2}_{\lambda;m}u; L_{2,\,\delta}(\mathbb{R}^n)\right\| \geq C\,\|u;\mathbb{H}_+\| \ \ (\lambda \geq \lambda_0 > 0)$$

for all $u \in \mathbb{H}_+$. This implies the invertibility of the operator $B^{1/2}_{\lambda;m}$ for $\lambda \geq \lambda_0 > 0$. Using [24, ch. 6, Theorem 3.2], we obtain the representation

$$A^{-1}_{\lambda;m} = B^{-1/2}_{\lambda;m}X_m(\lambda)B^{-1/2}_{\lambda;m} \ \ (\lambda \geq \lambda_0 > 0), \tag{3.50}$$

where $X_m(\lambda) : L_{2,\,\delta}(\mathbb{R}^n) \to L_{2,\,\delta}(\mathbb{R}^n)$ is some bounded operator, and its norm $\|X_m(\lambda)\|$ is not exceeds the numbers $M_1 > 0$, independent of $\lambda \in [\lambda_0,\infty)$.

Lemma 3. *There is a positive function $\omega_1(\lambda), \lambda > 0$, such that*

$$|\mathbb{K}_\lambda[F,\,v]| \leq \omega_1(\lambda)\,\|F;\mathbb{H}_-\|\cdot\|v;\mathbb{H}_+\| \tag{3.51}$$

for all $F \in L_{2,\,\delta}(\mathbb{R}^n)$, $v \in \mathbb{H}_+$, and $\omega_1(\lambda) \to 0$ for $\lambda \to \infty$.

Proof. We rewrite equality (3.31) in the form

$$\mathbb{K}_\lambda[F,\,v] =$$

$$= \sum_{m=1}^{\infty}\exp(i\theta_m)\sum\nolimits^{(1)} C^{k''}_{k'}\left(d^{\alpha_j}a_{klm}\varphi^{(k')}_m U^{(k'')}_{m,\lambda},\, d^{\alpha_j}v^{(l)}\right), \tag{3.52}$$

where

$$U_{m,\lambda}(x) = (\mathcal{R}_m(\lambda)\Phi_m F)(x), \quad m = 1,\,2,\,\ldots,$$

and the symbol $\sum^{(1)}$ denotes summation over multi-indices $k,\,l,\,k',\,k''$ such that $k = k' + k''$, $k' \neq 0$, $|k| = |l| = j \in J$.

Let $F \in L_{2,\delta}(\mathbb{R}^n)$. Using equalities (3.29), (3.49) – (3.52), we have

$$\mathbb{K}_\lambda[F, v] =$$

$$= \sum_{m=1}^{\infty} \sum\nolimits^{(1)} C_{k'}^{k''} \exp(i\theta_m) \left(d^{-|k|} a_{klm} \varphi_m^{(k')} D^{k''} A_{\lambda;m}^{-1} \Phi_m F,\, d^{-|l|} v^{(l)}\right)_{\alpha_j + j} =$$

$$= \sum_{m=1}^{\infty} \sum\nolimits^{(1)} C_{k'}^{k''} \exp(i\theta_m) \times$$

$$\times (d^{-|k|} a_{klm} \varphi_m^{(k')} D^{k''} B_{\lambda;m}^{-1/2} X_m(\lambda) B_{\lambda;m}^{-1/2} \Phi_m F, d^{-|l|} v^{(l)}(x))_{\alpha_j + j}.$$

Next, applying [22, Lemma 2.2] and the Cauchy-Bunyakovsky inequality, we have

$$|\mathbb{K}_\lambda[F, v]| \ll \Lambda_n \sup_{m=1,2,\dots} \sum\nolimits^{(3)} \left\|\mathbb{T}_{l,k',k'',m}(\lambda) V_{m,\lambda}; L_{2,\,\alpha_j + j}(\mathbb{R}^n)\right\| \times$$

$$\times \left\| d^{-|l|} v^{(l)}(x); L_{2,\,\alpha_j + j}(\mathbb{R}^n) \right\|, \quad (3.53)$$

where

$$\mathbb{T}_{l,k',k'',m}(\lambda) = d^{-|k'|-|k''|} \varphi_m^{(k')} a_{klm} D^{k''} B_{\lambda;m}^{-1/2},$$

$$V_{m,\lambda}(x) = X_m(\lambda) B_{\lambda;m}^{-1/2} (\varphi_m F)(x), \quad (3.54)$$

and the symbol $\sum^{(3)}$ denotes summation over multi-indices k', k'', l such that $|k'| + |k''| = |l| = j \in J$ and $k' \neq 0$.

Because

$$\left\| d^{-|l|} v^{(l)}(x); L_{2,\,\alpha_j + j}(\mathbb{R}^n) \right\| \leq \|v; \mathbb{H}_+\| \quad (v \in C_0^\infty(\mathbb{R}^n))$$

for any multi-index $l: \ |l| = j \in J$, then from (3.53) the inequality follows

$$|\mathbb{K}_\lambda[F, v]| \ll$$

$$\ll \|v; \mathbb{H}_+\| \sup_{m=1,2,\dots} \sum\nolimits^{(3)} \left\| \mathbb{T}_{l,k',k''m}^{k'}(\lambda) V_{m,\lambda}; L_{2,\,\alpha_j + j}(\mathbb{R}^n) \right\|, \quad (3.55)$$

Let λ_0 be the same number as in (3.37). Then for $\lambda > \lambda_0$ due to equality (3.36) we have

$$\left\| B_{\lambda;m}^{1/2} u; L_{2,\delta}(\mathbb{R}^n) \right\|^2 = \mathrm{Re}\{\exp(i\theta_m) \mathcal{B}_{\lambda;m}[u, u]\} =$$

$$= \mathrm{Re}\{\exp(i\theta_m) \mathcal{B}_{\lambda_0;m}[u, u]\} + \cos(\theta_m)(\lambda - \lambda_0) \int d^{2\delta}(x) |u(x)|^2 dx \geq$$

$$\geq \mathrm{Re}\{\exp(i\theta_m) \mathcal{B}_{\lambda_0;m}[u, u]\} = \left\| B_{\lambda_0;m}^{1/2} u; L_{2,\delta}(\mathbb{R}^n) \right\|^2. \quad (3.56)$$

It follows that for $\lambda \geq \lambda_0$ the operator $B_{\lambda_0;m}^{1/2} B_{\lambda;m}^{-1/2}$ is a bounded operator and its norm does not exceed one. Since the operator $B_{\lambda;m}$ is self-adjoint for all values of

$\lambda \geq 1$, then for $\lambda \geq \lambda_0$ the operator $B^{-1/2}_{\lambda;m} B^{1/2}_{\lambda_0;m}$ is also a bounded operator, and its norm does not exceed one. Taking this into account, we have

$$\left\| B^{-1/2}_{\lambda;m}(\varphi_m F); L_{2,\,\delta}(\Omega)\right\| \leq \leq \left\| B^{-1/2}_{\lambda;m} B^{1/2}_{\lambda_0;m}\right\| \times \left\| B^{-1/2}_{\lambda_0;m}(\varphi_m F); L_{2,\,\delta}(\Omega)\right\| \ll \ll \left\| B^{-1/2}_{\lambda_0;m}(\varphi_m F); L_{2,\,\delta}(\Omega)\right\|. \quad (3.57)$$

Let Ω be some region in $\mathbb{R}^n$. The norm in the space $L_2(\Omega)$ can be specified using the equality

$$\|f; L_2(\Omega)\| = \sup |(f, v)|, \quad (3.58)$$

where the supremum is taken over all $v \in L_2(\Omega)$ such that $\|v; L_2(\Omega)\| = 1$. Since $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$, then in equality (3.58) we can assume that the supremum is taken over all $v \in C_0^\infty(\Omega)$, such that $\|v; L_2(\Omega)\| = 1$.

In equality (3.58) we replace f with $d^\delta f$ and v with $d^\delta v$ and get the equality

$$\|f; L_{2,\,\delta}(\Omega)\| = \sup |(f, v)_\delta|, \quad (3.59)$$

where the supremum is taken over all $v \in C_0^\infty(\Omega)$ such that $\|v; L_{2,\,\delta}(\Omega)\| = 1$.

For $\lambda = \lambda_0$, from equality (3.34) we have $\left(B^{1/2}_{\lambda_0;m} u, B^{1/2}_{\lambda_0;m} v\right)_\delta = \widetilde{\mathcal{B}}_{\lambda_0;m}[u, v]$, On the other side,

$$\widetilde{\mathcal{B}}_{\lambda_0;m}[u, u] \gg \|u; \mathbb{H}_+\|^2,$$

$$\left|\widetilde{\mathcal{B}}_{\lambda_0;m}[u, v]\right| \leq (M_0 + \lambda_0)\, \|u; \mathbb{H}_+\| \cdot \|v; \mathbb{H}_+\|$$

for all $u, v \in C_0^\infty(\mathbb{R}^n)$. Therefore, according to the Lax-Milgram theorem, the equation $\widetilde{\mathcal{B}}_{\lambda_0;m}[u, \widehat{v}] = (w, \widehat{v})_\delta \ \ \forall \widehat{v} \in C_0^\infty(\mathbb{R}^n)$ has a solution for any $w \in L_{2,\,\delta}(\mathbb{R}^n)$. Therefore, the function $v \in C_0^\infty(\mathbb{R}^n)$ in (3.59) can be represented as $v = B^{1/2}_{\lambda_0;m} w$, that is $\|f; L_{2,\,\delta}(\mathbb{R}^n)\| = \sup \left|\left(f, B^{1/2}_{\lambda_0;m} w\right)_\delta\right|$, where the supremum is taken over all $w \in C_0^\infty(\mathbb{R}^n)$ such that $\left\|B^{1/2}_{\lambda_0;m} w; L_{2,\,\delta}(\mathbb{R}^n)\right\| = 1$.

On the other hand, in class $C_0^\infty(\mathbb{R}^n)$ the norms $\|v; \mathbb{H}_+\|$ and $\left\|B^{1/2}_{\lambda_0;m} v; L_{2,\,\delta}(\mathbb{R}^n)\right\|$ are equivalent. Therefore

$$\left\| B^{-1/2}_{\lambda_0;m}(\varphi_m F); L_{2,\,\delta}(\mathbb{R}^n)\right\| = \sup \left|\left(B^{-1/2}_{\lambda_0;m}(\varphi_m F), w\right)_\delta\right| = = \sup \left|\left(B^{-1/2}_{\lambda_0;m}(\varphi_m F), B^{1/2}_{\lambda_0;m} v\right)_\delta\right| \ll \sup |(\varphi_m F, v)_\delta| \ll \ll \|\varphi_m F; \mathbb{H}_-\| \ll \|F; \mathbb{H}_-\|, \quad (3.60)$$

where the first supremum is taken over all $w \in C_0^\infty(\mathbb{R}^n) : \|w; L_{2,\,\delta}(\mathbb{R}^n)\| = 1$, the second supremum is over all $v \in C_0^\infty(\mathbb{R}^n) : \left\|B^{1/2}_{\lambda_0;m} w; L_{2,\,\delta}(\mathbb{R}^n)\right\| = 1$, and the third supremum is over all $v \in C_0^\infty(\mathbb{R}^n) : \|v; \mathbb{H}_+\| = 1$.

Due to (3.57) from (3.60) the inequality follows $\|V_{m,\lambda}; L_{2,\,\delta}(\mathbb{R}^n)\| \leq M\, \|F; \mathbb{H}_-\|$, which is valid for $\lambda \geq \lambda_0$, where $\lambda_0 \geq 1$ is some finite number.

Since $\delta \le \alpha_j + j$ for all $j \in J$, then $d^{\alpha_j+j}(x) \le d^{\delta}(x)$ and from the resulting inequality follows that

$$\left\| V_{m,\lambda}; L_{2,\,\alpha_j+j}(\mathbb{R}^n)\right\| \le M \, \|F; \mathbb{H}_-\| \ \ (u \in \mathbb{H}_+,\, j \in J,\, \lambda \ge \lambda_0)\,. \tag{3.61}$$

From statement B) of Lemma 2 (see (3.35)) and equality (3.54) it follows that

$$\lim_{\lambda\to\infty} \|\mathbb{T}_{l,k',k''m}(\lambda)\| = 0. \tag{3.62}$$

In view of this equality, from (3.55), (3.61), (3.60) we obtain

$$|\mathbb{K}_\lambda[F, v]| \ll$$
$$\ll \sup_{m=1,\,2,\,\ldots} \ \sup_{|k'|+|k''|=j\in J;\, k'\neq 0} \|\mathbb{T}_{l,k',k'',m}(\lambda)\| \cdot \|V_{m,\lambda}; L_{2,\,\delta}(\mathbb{R}^n)\| \, \|v; \mathbb{H}_+\| \le$$
$$\le \omega_1(\lambda)\, \|F; \mathbb{H}_-\| \cdot \|v; \mathbb{H}_+\|$$

for all $F \in L_{2,\,\delta}(\mathbb{R}^n)$, $v \in \mathbb{H}_+$, and $\omega_1(\lambda) \to 0$ at $\lambda \to \infty$.

Thus, the estimate (3.51) is proven, which completes the proof of Lemma 3. □

Lemma 4. *There is a positive function $\omega_2(\lambda), \lambda > 0$, such that*

$$|\mathbb{L}_\lambda[F,\, v]| \le \omega_2(\lambda_0)\, \|F; \mathbb{H}_-\| \cdot \|v; \mathbb{H}_+\| \tag{3.63}$$

for all $\lambda \ge \lambda_0$ and for all $F \in L_{2,\,\delta}(\mathbb{R}^n)$, $v \in \mathbb{H}_+$. A positive function $\omega_2(\lambda_0)$ such that $\omega_2(\lambda_0) \to 0$ for $\lambda_0 \to \infty$.

Proof. We represent the sesquilinear form (see (3.32)) $\mathbb{L}_\lambda[F,\, v]$ in the form

$$\mathbb{L}_\lambda[F,\, v] = \sum_{m=1}^{\infty} \exp(i\theta_m) \sum_{|k|=|l|=j\in J} \sum\nolimits_l^{(2)} \mathbb{I}^{l',l''}_{\lambda;\,k,m}[F,\, v], \tag{3.64}$$

where

$$\mathbb{I}^{l',l''}_{\lambda;\,k,m}[F,\, v] = C^{l''}_{l'} \left(d^{-|k|} a_{klm} U^{(k)}_{m,\lambda},\, d^{-|l'|-|l''|} \varphi^{(l')}_m v^{(l'')} \right)_{\alpha_j+j}.$$

Since (see (3.49), (3.50))

$$\mathcal{R}_m(\lambda) = A^{-1}_{\lambda;m} = B^{-1/2}_{\lambda;m} X_m(\lambda) B^{-1/2}_{\lambda;m} \ \ (\lambda \ge \lambda_0 > 0),$$

then the form $\mathbb{I}^{l',l''}_{\lambda;\,k,m}[F,\, v]$ can be written as

$$\mathbb{I}^{l',l''}_{\lambda;\,k,m}[F,\, v] = \left(d^{-|k|} a_{klm} D^k B^{-1/2}_{\lambda;m} X_m(\lambda) B^{-1/2}_{\lambda;m} \Phi_m F,\, d^{-|l'|-|l''|} \varphi^{(l')}_m D^{l''} v \right)_{\alpha_j+j}.$$

Further, using the notation (3.54) we have

$$\mathbb{I}^{l',l''}_{\lambda;\,k,m}[F,\, v] = \left(d^{-|k|} a_{klm} D^k B^{-1/2}_{\lambda;m} V_{m,\lambda},\, d^{-|l'|-|l''|} \varphi^{(l')}_m D^{l''} v \right)_{\alpha_j+j}.$$

Since $D^{l''} v = D^{l''} B^{-1/2}_{\lambda_0;m} B^{1/2}_{\lambda_0;m} v$ and $B_{\lambda_0;m}$ is a self-adjoint operator, then

$$\mathbb{I}^{l',l''}_{\lambda;\,k,m}[F,\, v] =$$
$$= \left(B^{-1/2}_{\lambda_0;m} D^{l''} \varphi^{(l')}_m d^{-|k|-|l'|-|l''|} a_{klm} D^k B^{-1/2}_{\lambda;m} V_{m,\lambda},\, B^{1/2}_{\lambda_0;m} v \right)_{\alpha_j+j}. \tag{3.65}$$

In the process of proving Lemma 3, we used the notation (see (3.54))

$$\mathbb{T}_{l,k',k'',m}(\lambda) = d^{-|k'|-|k''|}\varphi_m^{(k')} a_{klm} D^{k''} B_{\lambda;m}^{-1/2}$$

and proved that (see (3.62))

$$\lim_{\lambda\to\infty} \|\mathbb{T}_{l,k',k'',m}(\lambda)\| = 0 \tag{3.66}$$

for $|k'| + |k''| = j \in J$; $k' \neq 0$. Hence,

$$\mathbb{T}_{k,l',l'',m}(\lambda_0) = d^{-|l'|-|l''|}\varphi_m^{(l')} a_{klm} D^{l''} B_{\lambda_0;m}^{-1/2}$$

and

$$\mathbb{T}^*{}_{k,l',l'',m}(\lambda_0) = B_{\lambda_0;m}^{-1/2} D^{l''} \varphi_m^{(l')} d^{-|l'|-|l''|} a_{klm}.$$

Since the norm of the bounded operator coincides with the norm of the conjugate operator, it follows from (3.66) that

$$\lim_{\lambda_0\to\infty} \|\mathbb{T}^*{}_{l',l'',m}(\lambda_0)\| = 0 \tag{3.67}$$

for $|l'| + |l''| = j \in J$; $l' \neq 0$.

Using the introduced notation, equality (3.65) is written in the form

$$\mathbb{I}^{l',l''}_{\lambda;\,k,m}[F,\,v] = \Big(\mathbb{T}^*{}_{l',l'',m}(\lambda_0) d^{-|k|} D^k B_{\lambda;m}^{-1/2} V_{m,\lambda},\; B_{\lambda_0;m}^{1/2} v\Big)_{\alpha_j+j}.$$

Future we introduce a notation

$$\mathbb{P}_{m,\lambda_0,k} = d^{-|k|} D^k B_{\lambda_0;m}^{-1/2}$$

and write the resulting equality in the form

$$\mathbb{I}^{l',l''}_{\lambda;\,k,m}[F,\,v] = \Big(\mathbb{T}^*{}_{k,l',l'',m}(\lambda_0) \mathbb{P}_{m,\lambda_0,k} B_{\lambda_0;m}^{1/2} B_{\lambda;m}^{-1/2} V_{m,\lambda},\; B_{\lambda_0;m}^{1/2} v\Big)_{\alpha_j+j}. \tag{3.68}$$

From (3.56) it follows that for $\lambda \geq \lambda_0$ the operator $B_{\lambda_0;m}^{1/2} B_{\lambda;m}^{-1/2}$ is a bounded operator and its norm does not exceed one. On the other hand, according to part A) of Lemma 2, the operator $\mathbb{P}_{m,\lambda_0,k}$ is bounded. Therefore, from (3.68) we have

$$\left|\mathbb{I}^{l',l''}_{\lambda;\,k,m}[F,\,v]\right| \leq$$

$$\leq M_2 \|\mathbb{T}^*{}_{k,l',l'',m}(\lambda_0)\| \left\|V_{m,\lambda};\, L_{2,\,\alpha_j+j}(\mathbb{R}^n)\right\| \cdot \left\|B_{\lambda_0;m}^{1/2} v;\, L_{2,\,\alpha_j+j}(\mathbb{R}^n)\right\| \tag{3.69}$$

for all $\lambda \geq \lambda_0$.

Previously, when proving Lemma 3, we proved that (see (3.61)),

$$\|V_{m,\lambda}; L_{2,\,\delta}(\mathbb{R}^n)\| \leq M_3 \|F; \mathbb{H}_-\|,$$

for $\lambda \geq \lambda_0$, where $\lambda_0 \geq 1$ is a finite number and

$$\left\|B_{\lambda_0;m}^{1/2} v;\, L_{2,\,\delta}(\mathbb{R}^n)\right\| \leq M_4 \|v; \mathbb{H}_+\|$$

for all $v \in \mathbb{H}_+$. Due to these inequalities, it follows from (3.69) that

$$\left|\mathbb{I}^{l',l''}_{\lambda;\,k,m}[F,\,v]\right| \leq M_5 \|\mathbb{T}^*{}_{k,l',l'',m}(\lambda_0)\| \|F; \mathbb{H}_-\| \cdot \|v; \mathbb{H}_+\|$$

for all $\lambda \geq \lambda_0$ and for all $F \in L_{2,\delta}(\mathbb{R}^n)$, $v \in \mathbb{H}_+$. Introducing a notation $\delta_*(\lambda_0) = M_5 \sup \|\mathbb{T}^*{}_{k,l',l'',m}(\lambda_0)\|$, we get

$$\left|\mathbb{I}^{l',l''}_{\lambda;\,k,m}[F,\,v]\right| \leq \delta_*(\lambda_0)\,\|F;\mathbb{H}_-\|\cdot\|v;\,\mathbb{H}_+\| \tag{3.70}$$

for all $\lambda \geq \lambda_0$ and for all $F \in L_{2,\delta}(\mathbb{R}^n)$, $v \in \mathbb{H}_+$.

From (3.67) it follows that $\delta_*(\lambda_0) \to 0$ for $\lambda_0 \to 0$. Therefore, choosing the number λ_0 large enough, from (3.70) and (3.64) by virtue of [22, Lemma 2.2] we obtain (3.63).

□

Applying the inequalities (3.51), (3.63), established, respectively, in Lemmas 3 and 4, from (3.30) we obtain

$$|\langle\mathbb{R}(\lambda)F,v\rangle - \langle F,\,v\rangle| \leq (\omega_1(\lambda)+\omega_2(\lambda_0))\,\|F;\mathbb{H}_-\|\cdot\|v;\mathbb{H}_+\|$$

for all $F \in L_{2,\delta}(\mathbb{R}^n)$, $v \in \mathbb{H}_+$. Since $\omega_1(\lambda) \to 0$ at $\lambda \to \infty$ and $\omega_2(\lambda_0) \to 0$ for $\lambda_0 \to \infty$, then there is a number $\lambda_0 \geq 1$ such that

$$|\langle\mathbb{R}(\lambda)F,v\rangle - \langle F,\,v\rangle| \leq \frac{1}{2}\,\|F;\mathbb{H}_-\|\cdot\|v;\mathbb{H}_+\| \tag{3.71}$$

for any $\lambda \geq \lambda_0$ and for all $F \in L_{2,\delta}(\mathbb{R}^n)$, $v \in \mathbb{H}_+$. Since, by the definition of the space $\mathbb{H}_-$, the space $L_{2,\delta}(\mathbb{R}^n)$ is densely embedded in $\mathbb{H}_-$, then the estimate (3.71) true for every $F \in \mathbb{H}_-$.

From the estimate (3.71) it follows that for $\lambda > \lambda_0$ the operator $\mathbb{G}(\lambda) = \mathbb{R}(\lambda) - E$, acting from $\mathbb{H}_-$ to $\mathbb{H}_-$ is bounded and its norm does not exceed 1/2. Therefore the operator $\mathbb{R}(\lambda) : \mathbb{H}_- \to \mathbb{H}_-$ is continuously invertible and $\mathbb{R}^{-1}(\lambda) = (E + \mathbb{G}(\lambda))^{-1}$.

The operator $\mathcal{R}_m(\lambda)$ acts from $\mathbb{H}_-$ to $\mathbb{H}_+$. Therefore, from (3.24) it follows that the operator $\mathcal{R}(\lambda)$ also acts from $\mathbb{H}_-$ to $\mathbb{H}_+$. Therefore, for any functional $F \in \mathbb{H}_-$ the function $U(x)$ defined by the equality

$$U = \mathcal{R}(\lambda)\mathbb{R}^{-1}(\lambda)F \quad (\lambda \geq \lambda_0), \tag{3.72}$$

belongs to the space $\mathbb{H}_+$.

Further we will assume that $\lambda \geq \lambda_0$ and λ_0 are some sufficiently large number. Then from equality (3.24) it follows that $B_\lambda[\mathcal{R}(\lambda)\mathbb{R}^{-1}(\lambda)F,v] = \langle F,v\rangle$ for all $v \in C_0^\infty(\mathbb{R}^n)$. Therefore, for $\lambda \geq \lambda_0$, the function $U(x)$ defined by the equality (3.72) satisfies the equality $B_\lambda[U,v] = \langle F,v\rangle \ \ \forall v \in C_0^\infty(\mathbb{R}^n)$. This means that the function (3.72) is a solution to the problem $\mathbb{D}_\lambda$. Since for $\lambda \geq \lambda_0$ operator $\mathbb{R}^{-1}(\lambda)$ is bounded, then from (3.23) and (3.24) it follows that the function (3.72) satisfies estimate (2.6) of the main Theorem.

Thus, we have proven that the problem $\mathbb{D}_\lambda$ for $\lambda \geq \lambda_0$ has a solution for any given functional $F \in \mathbb{H}_-$, and it satisfies the inequality (2.6).

Now we proceed to the proof of the uniqueness of the solution to the problem $\mathbb{D}_\lambda$.

Consider the conjugate problem: for a given functional $F \in \mathbb{H}_-$ find the function $U_1 \in \mathbb{H}_+$ satisfying the equality

$$\overline{B_\lambda[v,U_1]} = \langle F,v\rangle \quad \forall v \in \mathbb{H}_+. \tag{3.73}$$

Since the coefficients of the bilinear form $\overline{B_\lambda[v, U_1]}$ satisfy the conditions of the theorem 1, proceeding in the same way as above, we can construct the operators $\mathcal{R}_*(\lambda)$, $\mathbb{R}_*(\lambda)$ such that the function $U_1 = \mathcal{R}_*(\lambda)\mathbb{R}_*(\lambda)^{-1}F$ $(\lambda \in [\lambda_0^*, \infty))$ belongs to the space $\mathbb{H}_+$ and satisfies the equation (3.73).

Let the function $u \in \mathbb{H}_+$ be a solution to the equation

$$B_\lambda[u, v] = 0 \quad (\forall v \in \mathbb{H}_+), \tag{3.74}$$

where $\lambda \geq \lambda_0' = \max\{\lambda_0^*, \lambda_0\}$. Let F be an arbitrary element of space $\mathbb{H}_-$. Since $U_1 = \mathcal{R}_*(\lambda)\mathbb{R}_*(\lambda)^{-1}F$ belongs to the space $\mathbb{H}_+$, then, putting $v = U_1$ in (3.74), we obtain $B_\lambda[u, U_1] = 0$, that is $\overline{B_\lambda[u, U_1]} = 0$. On the other hand, the function $U_1 = \mathcal{R}_*(\lambda)\mathbb{R}_*(\lambda)^{-1}F$ satisfies (3.73). Therefore $\langle F, u\rangle = 0$ for all $F \in \mathbb{H}_+$. Taking into account the embedding $\mathbb{H}_+ \to \mathbb{H}_-$ and setting $F = u$, we have $\langle u, u\rangle = 0$, that is, $u = 0$.

The main theorem is completely proven.

References

1. Nikol'skii, S.M. A variational problem for an equation of elliptic type with degeneration on the boundary, in: *Proc. Steklov Inst. Math.* **150** (1981), 227–254.
2. Lizorkin, P.I., Nikol'skij, S.M.: Coercive properties of elliptic equations with degeneration. Variational method, in: *Number theory, mathematical analysis and their applications. Trudy Mat. Inst. Steklov* **157** (1981) 90–118.
3. Lizorkin, P.I., Nikol'skij, S.M.: Coercive properties of an elliptic equation with degeneration and a generalized right-hand side, in: *Proc. Steklov Inst. Math.* **161** (1984) 171–198.
4. Baidel'dinov, B.L.: An analogue of the first boundary value problem for elliptic equations with degeneration. The method of bilinear forms, in: *Proc. Steklov Inst. Math.* **170** (1987), 1–10.
5. Lizorkin, P.I.: On the theory of degenerate elliptic equations, in: *Proc. Steklov Inst. Math.* **172** (1985) 257–274
6. Miroshin, N.V.: The variational Dirichlet problem for an elliptic operator that is degenerate on the boundary, in: *Differ. Equ.* **24** (1988), no.3, 323-329
7. Nikol'skii, S.M., Lizorkin, P.I., Miroshin, N.V.: Weighted function spaces and their applications to the investigation of boundary value problems for degenerate elliptic equations, in: *Soviet Math. (Iz. VUZ)* **32** (1988), no. 8, 1–40.
8. Iskhokov, S.A., Kudzhmuratov, A.Ya.: On the variational Dirichlet problem for degenerate elliptic operators, in: *Doklady Mathematics* **72** (2005), no. 1, 512–515.
9. Boimatov, K.Kh.: The generalized Dirichlet problem for systems of second-order differential equations, in: *Dokl. Math.* **46** (1993), no. 3, 403–409.
10. Boimatov, K.Kh.: The generalized Dirichlet problem associated with a noncoercive bilinear form, in: *Dokl. Math.* **47** (1993), no. 3, 455–463.
11. Boimatov, K.Kh.: Matrix differential operators generated by noncoercive bilinear forms, in: *Dokl. Math.* **50** (1995), no. 3, 351–359.
12. Boimatov K.Kh.: Boundary value problems for noncoercive forms, in: *Reports of Academy of sciences of the Republic of Tajikistan* **XLI** (1998), no. 10, 10–16.
13. Boimatov, K.Kh.: On the Abel basis property of the system of root vector-functions of degenerate elliptic differential operators with singular matrix coefficients, in: *Siberian Math. J.* **47** (2006), no. 1, 35–44.
14. Boimatov, K.Kh., Iskhokov, S.A.: On the solvability and smoothness of a solution of the variational Dirichlet problem associated with a noncoercive bilinear form, in: *Proc. Steklov Inst. Math.* **214** (1996), 101–127.
15. Iskhokov, S.A.: On the smoothness of a solutions of the generalized Dirichlet problem and the eigenvalue problem for differential operators generated by noncoercive bilinear form, in: *Doklady Mathematics* **51** (1995), no. 3, 323–325.

16. Iskhokov, S.A.: The variational Dirichlet problem for elliptic operators with nonpower degeneration generated by noncoercive bilinear forms, in: *Doklady Mathematics* **68** (2003), no. 2, 261–265.
17. Iskhokov, S.A., Yakushev I.A.: On solvability of variational Dirichlet problem for a class of degenerate elliptic operators in: *Chebyshevskii Sbornik* **19** (2018), no. 3, 164–182.
18. Iskhokov, S.A., Gadoev, M.G., Petrova, M.I.: On some spectral properties of a class of degenerate elliptic differential operators, in: *Mathematical notes of SVFU* **23** (2016), no. 2, 31–50.
19. Iskhokov, S.A, Rakhmonov B.A.: On the solvability and smoothness of the solution to the variational Dirichlet problem for degenerate elliptic operators in the whole space, in: *Reports of Academy of sciences of the Republic of Tajikistan* **61**, (2018), no. 3, 224–230.
20. Iskhokov, S.A., Rakhmonov, B.A.: Variational Dirichlet problem associated with a noncoercive form in the whole space, in: *Proceedings of the Academy of Sciences of the Republic of Tajikistan. Dept. Phys.-Math., Chem., Geol. and tech. sc.*, no. 2(171) (2018) 17–25.
21. Iskhokov, S.A., Rakhmonov, B.A.: Solvability and smoothness of solution to variational Dirichlet problem in entire space associated with a non-coercive form, in: *Ufa Math. J.* **12**, (2020), no. 1, 13–29.
22. Boimatov, K.Kh.: Separability theorems, weighted spaces and their applications, in: *Proc. Steklov Inst. Math.* **170** (1987), 39–81.
23. Iskhokov, S.A.: Garding's Inequality for Elliptic Operators with Degeneracy, in: *Math. Notes* **87** (2010), no. 2, 189—203.
24. Kato, T.: *Perturbation Theory for Linear Operators*, Springer Berlin, Heidelberg, 1995.

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