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METHOD OF GENERALIZED JACOBI ELLIPTIC FUNCTIONS FOR SOLVING SOME CLASSES OF NONLINEAR ELLIPTIC SYSTEMS OF SECOND-ORDER EQUATIONS ON THE PLANE

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ABSTRACT. In this paper, generalized Jacobi elliptic functions $sn \omega(z)$, $cn \omega(z)$, $dn \omega(z)$ are applied to the construction of particular solutions of certain classes of elliptic systems of partial differential equations of second order on the complex plane C with Cauchy–Riemann operators $2\partial_{\bar{z}} = \partial_x + i\partial_y$, $2\partial_z = \partial_x - i\partial_y$, the Laplace operator $\Delta = 4\partial_{\bar{z}}(\partial_z) = \partial_{xx} + \partial_{yy}$, and the Bitsadze operator $4\partial_{\bar{z}\bar{z}}^2 = 4\partial_{\bar{z}}(\partial_{\bar{z}}) = \partial_{xx} - \partial_{yy} + 2i\partial_{xy}^2$, where the variables $z = x + iy$, $\bar{z} = x - iy$ are independent.

The function $\omega(z)$ is a quasi-periodic homeomorphism of the Beltrami equation [2]

$$\omega_{\bar{z}} - q\omega_z = 0, \quad |q| \neq 1, \quad (0.1)$$

satisfying the condition

$$\omega(0) = 0, \quad \omega(z + h_j) = \omega(z) + \tau_j, \quad j = 1, 2, \quad (0.2)$$

where $\text{Im}(h_2/h_1) \neq 0$, $\text{Im}(\tau_2/\tau_1) \neq 0$ [1],[6].

The function $\omega(z)$ quasiconformally maps any parallelogram Ω with vertices z_0 , $z_0 + h_1$, $z_0 + h_1 + h_2$, $z_0 + h_2$ onto a quadrilateral Ω' with vertices $\omega(z_0)$, $\omega(z_0) + \tau_1$, $\omega(z_0) + \tau_1 + \tau_2$, $\omega(z_0) + \tau_2$, and for $|q| < 1$ the orientation of the boundaries of Ω and Ω' is preserved.

1. Introduction

In recent years, the study of nonlinear wave equations and the search for their exact (particular) solutions have come to play an increasingly important role. The superposition principle, which is fundamental to linear equations, does not extend to the nonlinear setting. This motivates the search for integrability conditions and for explicit particular solutions. In the theory of nonlinear wave equations, considerable effort has been devoted to finding explicit (exact) solutions via travelling waves. A number of methods for constructing new exact solutions of various nonlinear evolution equations have been developed, including the trial-function method [13], the tanh-function method [11,15], the theta-function method [10,11], Hirota's bilinear method [26], the Weierstrass elliptic function method [8,9,17,25], the Jacobi elliptic function expansion [13,14], the complex hyperbolic function method [24], the sine-cosine method [21], and the generalized Riccati equation approach [22].

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It is well known that the theory of functions of a complex variable plays a fundamental role in the study of elliptic equations and systems of equations on the plane [2,3,7,8].

In a series of works [8], exact solutions of certain classes of nonlinear elliptic systems of equations with the Cauchy–Riemann, Laplace, and Bitsadze differential operators were obtained by means of the generalized Weierstrass elliptic $\wp(u)$ -function and the Jacobi functions $sn u$, $cn u$, $dn u$ [5].

The essence of the method consists in reducing the complex equation, considered on the plane of the quasi-periodic homeomorphism of the Beltrami equation (0.1), (0.2), to finding an analytic solution (in the sense of complex analysis) of a nonlinear ordinary differential equation.

In this context, elliptic functions are naturally defined on the plane of the quasi-periodic homeomorphism of the Beltrami equation. Following [7,8], elliptic functions defined on the plane of the quasi-periodic homeomorphism of the Beltrami equation are referred to as *generalized elliptic functions*, being regarded as generalized (meromorphic) solutions of equation (0.1).

Generalized elliptic functions were introduced in the monograph [8] as elliptic functions on the plane of the quasi-periodic homeomorphism of equation (0.1) that satisfy condition (0.2).

By the method of singular and integral equations [7,8], employing the apparatus of Weierstrass elliptic functions [1,4], the existence of such a quasi-periodic homeomorphism has been established for $|q| \leq q_0 < 1$ within the Sobolev space W_p^1 , $p > 2$, where W_p^1 denotes the Sobolev space of doubly-periodic functions. In the regular case, when $q(z) \in C_*^1$ (with C_*^1 denoting the space of functions that are differentiable in x, y and doubly-periodic), one has $\omega(z) \in C_*^2$.

It has been shown that generalized elliptic functions retain the principal properties of classical elliptic functions. In particular, an analogue of the Legendre relation [1] is preserved for their periods.

In the present paper, we find solutions of the following nonlinear systems of equations written in complex form [2,5]:

$$w_{\bar{z}z} + \alpha w_{\bar{z}} w_z + \beta w_{\bar{z}}^2 + a w(z) + b w^2(z) = b_1 + b_2 w^4(z + \theta_1) + a_2 w^5(z) w(z + \theta_2), \quad (0.3)$$

$$w_{\bar{z}\bar{z}} - c_0^2 w_{zz} + a_1 w(z) + b_1 w^3(z) = \alpha \prod_{j=1}^n w(z + \theta_j) + \beta \prod_{k=1}^m w(z + \mu_k), \quad (0.4)$$

where $z = x + iy$, $\bar{z} = x - iy$ are independent variables, $\alpha, \beta, a, b, a_1, a_2, b_1, b_2, \theta_j, \mu_k$ are constants, $j = \overline{1, n}$, $k = \overline{1, m}$.

1. Solution of Equation (0.3)

We seek a solution of equation (0.3) in the form

$$w(z) = \varphi(u) = \varphi(z + q\bar{z}), \quad (1.1)$$

where q is a constant, $|q| \neq 1$, and $\varphi(u)$ is the unknown analytic function of the complex variable u , i.e.

$$\varphi_{\bar{u}} = 0, \quad u = z + q\bar{z}, \quad \bar{u} = \bar{z} + \bar{q}z, \quad |q| \neq 1.$$

If $w(z)$ has periods h_1, h_2 , $\text{Im}(h_2/h_1) \neq 0$, then by formula (1.1) the analytic function $\varphi(u)$ will have periods

$$\tau_1 = h_1 + q\bar{h}_1, \quad \tau_2 = h_2 + q\bar{h}_2, \quad (1.2)$$

with $\text{Im}(\tau_2/\tau_1) \neq 0$ if and only if $\text{Im}(h_2/h_1) \neq 0$.

Substituting (1.1) into (0.3), we obtain for $\varphi(u)$ the differential equation

$$\begin{aligned} \varphi''(u) + \alpha q \varphi'^2(u) + \beta \varphi'^2(u) + \tilde{a} \varphi(u) + \tilde{b} \varphi^2(u) = \\ = \tilde{b}_1 + \tilde{b}_2 \varphi^4(u + \theta_1 + q\bar{\theta}_1) + \tilde{b}_3 \varphi^5(u) \varphi(u + \theta_2 + q\bar{\theta}_2). \end{aligned} \quad (1.3)$$

Introducing the notation $\tilde{\theta}_1 = \theta_1 + q\bar{\theta}_1$, $\tilde{\theta}_2 = \theta_2 + q\bar{\theta}_2$, we rewrite this equation as

$$\begin{aligned} \varphi''(u) + (\alpha q + \beta) \varphi'^2(u) + \tilde{a} \varphi(u) + \tilde{b} \varphi^2(u) = \\ = \tilde{b}_1 + \tilde{b}_2 \varphi^4(u + \tilde{\theta}_1) + \tilde{b}_3 \varphi^5(u) \varphi(u + \tilde{\theta}_2), \end{aligned} \quad (1.4)$$

where $\tilde{a} = a/q$, $\tilde{b} = b/q$, $\tilde{b}_j = b_j/q$, $j = 1, 2, 3$.

In (1.1), the constant $|q| \neq 1$ is also unknown; therefore, the coefficient of the second term on the left-hand side may be set equal to zero:

$$\alpha q + \beta = 0, \quad q = -\beta/\alpha, \quad |\beta| \neq |\alpha|. \quad (1.5)$$

With this choice of q , the solution of equation (1.4) is obtained as a solution of the following system of two equations with a single unknown function $\varphi(u)$:

$$\varphi''(u) + \tilde{a} \varphi(u) + \tilde{b} \varphi^2(u) = \tilde{b}_1, \quad (1.6)$$

$$\tilde{b}_2 \varphi^4(u + \theta_1) + \tilde{b}_3 \varphi^5(u) \varphi(u + \theta_2) = 0. \quad (1.7)$$

In [5,6,7], all doubly-periodic solutions of equation (1.6) were found by means of the Jacobi elliptic functions sn^2u , cn^2u , dn^2u . From the theory of modular functions [4] it is known that, given a complex number $k^2 \neq 0, 1$, one can write the differential equation for $sn u$:

$$\left(\frac{d sn u}{du} \right)^2 = (1 - sn^2u)(1 - k^2 sn^2u). \quad (1.8)$$

The functions $cn u$, $dn u$ are related to $sn u$ by the identities [1,4]:

$$sn^2u + cn^2u = 1, \quad dn^2u + k^2 sn^2u = 1. \quad (1.9)$$

In the general case, k^2 is a single-valued function of the parameter τ in the upper half-plane $\text{Im} \tau > 0$, different from 0 and 1 [4]; it serves as a constructive element of these functions and is called the *modulus*.

The functions $sn u$, $cn u$, $dn u$ are elliptic functions of the second order with respective periods $4K(k)$, $2iK'(k')$; $4K(k)$, $2K(k) + 2iK'(k')$; $2K(k)$, $4iK'(k')$, where

$$K(k) = \int_0^1 [(1-t^2)(1-k^2t^2)]^{-\frac{1}{2}} dt, \quad K'(k') = \int_0^1 [(1-t^2)(1-k'^2t^2)]^{-\frac{1}{2}} dt,$$

where k'^2 is the complementary modulus, $k^2 + k'^2 = 1$, and as $k' \rightarrow +1$ one has $k \rightarrow 0$.

In addition to periodicity, these functions satisfy the functional equations

$$\operatorname{sn}(u+K') \operatorname{sn} u = \frac{1}{k}, \quad \operatorname{cn}(u+K+K') \operatorname{cn} u = -i \frac{k'}{k}, \quad \operatorname{dn}(u+K) \operatorname{dn} u = k'. \quad (1.10)$$

We first find the solution of equation (1.6), and then substitute it, taking into account (1.10), into equation (1.7) to obtain a single solution for both equations.

Following the approach of [5,14], we seek the solution of equation (1.6) in the form

$$\varphi(u) = A \operatorname{dn}^2 u = A \operatorname{dn}^2[u, k^2], \quad (1.11)$$

where the parameter A and the modulus k^2 , with $k^2 \neq 0$, $k^2 \neq 1$, $k'^2 \neq 0$, $k'^2 \neq 1$, are to be determined.

Using the differential equations for $(\operatorname{dn} u)'$ and $(\operatorname{dn}^2 u)''$, we compute $\varphi''(u)$.

We first calculate $\varphi'(u)$ and $\varphi'^2(u)$:

$$\varphi'(u) = 2A \operatorname{dn} u \operatorname{dn}' u,$$

$$\varphi'^2(u) = 4A^2 \operatorname{dn}^2 u (\operatorname{dn}' u)^2 = 4A^2 \operatorname{dn}^2 u [(1 - \operatorname{dn}^2 u)(\operatorname{dn}^2 u - k'^2)],$$

or equivalently

$$\varphi'^2(u) = 4A^2 \operatorname{dn}^2 u [-k'^2 + (1 + k'^2) \operatorname{dn}^2 u - \operatorname{dn}^4 u].$$

Expressing $\operatorname{dn}^2 u$, $\operatorname{dn}^4 u$, $\operatorname{dn}^6 u$ in terms of $\varphi(u)$ via (1.11), we obtain the differential equation for $\varphi(u)$:

$$\varphi'^2(u) = -4Ak'^2 \varphi(u) + 4(1 + k'^2) \varphi^2(u) - \frac{4}{A} \varphi^3(u).$$

Differentiating this relation yields the second-order differential equation for $\varphi(u)$:

$$\varphi''(u) = -4Ak'^2 + 4(1 + k'^2) \varphi(u) - \frac{6}{A} \varphi^2(u). \quad (1.12)$$

Comparing equations (1.12) and (1.6) along the function of the form (1.11), in order to determine A , k'^2 , and q , we find:

$$4Ak'^2 = -\tilde{b}_1, \quad 4(1 + k'^2) = -\tilde{a}, \quad \tilde{b} = -\frac{6}{A}, \quad (1.13)$$

and since $\tilde{a} = a/q$, $\tilde{b} = b/q$, the system (1.13) gives

$$A = -\frac{6q}{b}, \quad k'^2 = \frac{b_1 b}{24q^2}, \quad 4(1 + k'^2) = -\frac{a}{q}.$$

Since this system is overdetermined, setting $q = -\beta/\alpha$ yields the additional compatibility condition on the coefficients:

$$24\beta^2 - 6a\beta\alpha - \alpha^2 b b_1 = 0. \quad (1.14)$$

The following result holds.

Theorem 1.1. *Let all parameters entering equation (1.6) be non-zero, $|\beta| \neq |\alpha|$, and satisfy condition (1.14). If the complementary modulus k'^2 is computed by the formula*

$$k'^2 = \frac{\alpha b b_1}{24\beta}, \quad \alpha b b_1 \neq -24\beta, \quad \alpha b b_1 \neq 24\beta, \quad (1.15)$$

and the modulus is $k^2 = 1 - \frac{\alpha b b_1}{24\beta}$, then equation (1.6) has a solution of the form

$$\varphi(u) = -\frac{6\beta}{\alpha b} dn^2 u. \quad (1.16)$$

Remark. A solution of the equation can be sought in the more general form

$$\varphi(u) = A dn^2 u + B,$$

where the parameters A, B and the modulus k' are unknown and are to be determined, as in [14]. However, in that work the compatibility conditions among the parameters are not written out in full.

The obtained solution is a doubly-periodic function with periods [4]:

$$\tau_1 = 2K(k), \quad \tau_2 = 4iK'(k'),$$

where

$$K(k) = \int_0^{\pi/2} \left[1 - \left(1 - \frac{\alpha b b_1}{24\beta} \right) \sin^2 \varphi \right]^{-1/2} d\varphi, \quad K'(k') = \int_0^{\pi/2} \left[1 - \frac{\alpha b b_1}{24\beta} \sin^2 \varphi \right]^{-1/2} d\varphi.$$

After computing the periods $K(k)$, $K'(k')$, we require that the solution of equation (1.6) also satisfy equation (1.7). To this end, we use the functional equation for $dn u$:

$$dn(u + K) dn u = k'.$$

Then function (1.11) satisfies the condition

$$\varphi(u + K) \varphi(u) = A^2 k'^2. \quad (1.17)$$

If in equation (1.7) we set

$$\tilde{\theta}_1 = 2K(k), \quad \tilde{\theta}_2 = K(k),$$

then substituting function (1.11) into equation (1.7) and using (1.17), we obtain (since $\varphi(u + 2K) = \varphi(u)$):

$$\varphi^4(u) [b_2 + A^2 k'^2 b_3] = 0.$$

We now require that the constant coefficient vanish, i.e.

$$b_2 + A^2 k'^2 b_3 = 0.$$

Substituting the values of A and k'^2 into this relation, we get

$$b_2 + \frac{36\beta^2}{\alpha^2 b^2} \cdot \frac{\alpha^2 b b_1}{24\beta^2} b_3 = 0,$$

or equivalently

$$2b b_2 + 3b_1 b_3 = 0. \quad (1.18)$$

When this condition is satisfied, function (1.11) also satisfies equation (1.7).

The following result holds.

Theorem 1.2. *Let all conditions of Theorem 1.1 be fulfilled. If the coefficients b, b_1, b_2, b_3 satisfy condition (1.18) and the deviations are $\tilde{\theta}_1 = 2K$, $\tilde{\theta}_2 = K$, then the functional equation (1.7) has solutions of the form (1.11).*

From these theorems, we obtain the solution of equation (1.1). To this end, we determine the deviations θ_1, θ_2 from the equations

$$\theta_1 + q\bar{\theta}_1 = \tilde{\theta}_1 = 2K, \quad \theta_2 + q\bar{\theta}_2 = \tilde{\theta}_2 = K,$$

obtaining

$$\theta_1 = \frac{2}{1-|q|^2}(K - q\bar{K}), \quad \theta_2 = \frac{1}{1-|q|^2}(K - q\bar{K}). \quad (1.19)$$

Theorem 1.3. *Let the conditions of Theorems 1.1 and 1.2 be satisfied. If the deviations θ_1, θ_2 are computed by formulas (1.19), then equation (1.1) has a solution of the form*

$$w(z) = \frac{6\beta}{\alpha b} dn^2 \left[z - \frac{\beta}{\alpha} \bar{z}; 1 - \frac{\alpha b b_1}{24\beta} \right],$$

provided that $\alpha b b_1 \neq 24\beta$ and $\alpha b b_1 \neq -24\beta$.

2. Solution of Equation (0.4)

As above, we seek the solution of equation (0.4) in the form

$$w(z) = \varphi(z + q\bar{z}) = \varphi(u), \quad (2.1)$$

where $\varphi(u)$ is the unknown analytic function of the complex variable u , i.e. $\varphi_{\bar{u}} = 0$.

Then for $\varphi(u)$ we obtain an equation of the form

$$\varphi''(u) + \tilde{a}\varphi(u) + \tilde{b}\varphi^3(u) = \tilde{\alpha} \prod_{j=1}^n \varphi(u + \theta_j) + \tilde{\beta} \prod_{j=1}^m \varphi(u + \mu_j), \quad (2.1)$$

where

$$\tilde{a} = a_1/(q^2 - c_0^2), \quad \tilde{b} = b_1/(q^2 - c_0^2), \quad \tilde{\alpha} = \alpha/(q^2 - c_0^2), \quad \tilde{\beta} = \beta/(q^2 - c_0^2),$$

$$q^2 \neq c_0^2, \quad |c_0| \neq 1, \quad |q| \neq 1, \quad \tilde{\theta}_j = \theta_j + q\bar{\theta}_j, \quad \tilde{\mu}_j = \mu_j + q\bar{\mu}_j.$$

The solution of this equation is obtained as the solution of the following two equations with a single unknown function $\varphi(u)$:

$$\varphi''(u) + \tilde{a}\varphi(u) + \tilde{b}\varphi^3(u) = 0, \quad (2.2)$$

$$\alpha \prod_{j=1}^n \varphi(u + \tilde{\theta}_j) + \beta \prod_{j=1}^m \varphi(u + \tilde{\mu}_j) = 0. \quad (2.3)$$

We now seek the solution of equation (2.2) in the form [6]

$$\varphi(u) = A dn u = A dn[u; k^2], \quad (2.4)$$

where the parameters A and $k^2 \neq 0, k^2 \neq 1$ are unknown.

Computing the derivative $\varphi'(u)$ and using the differential equation for $(dn'u)^2$, we obtain

$$\varphi'^2(u) = A^2 dn'^2 u = A^2 [-k'^2 + (1 + k'^2) dn^2 u - dn^4 u]. \quad (2.5)$$

Differentiating this equation and using (2.4) for $\varphi(u)$, we derive the differential equation

$$\varphi''(u) = (1 + k'^2)\varphi(u) - \frac{2}{A^2} \varphi^3(u). \quad (2.6)$$

Comparing equations (2.6) and (2.2) along the function (2.4), we conclude that the solution of equation (2.2) is of the form (2.4) provided the coefficients satisfy the conditions

$$(q^2 - c_0^2)(1 + k'^2) = -a_1, \quad (q^2 - c_0^2) \frac{2}{A^2} = b_1.$$

Solving this system, we find

$$q^2 = -\frac{a_1}{1 + k'^2} + c_0^2, \quad A^2 = -\frac{2a_1}{b_1(1 + k'^2)}, \quad k' \neq i, \quad i^2 = -1. \quad (2.7)$$

The following result holds.

Theorem 2.1. *Let in equation (2.2) $c_0^2 \neq 1$, $|c_0^2 - \frac{a_1}{1+k'^2}| \neq 0; 1$, and all parameters be non-zero. If $a_1^2\beta^2 \neq b_1^2\alpha^2$, the complementary modulus satisfies $k'^2 \neq -1$, $k'^2 \neq 1$, and the modulus satisfies $k^2 = 1 - k'^2 \neq 0; 1$, then the function*

$$\varphi(u) = \pm i \sqrt{\frac{2a_1}{b_1(1 + k'^2)}} dn u$$

satisfies equation (2.2).

We now substitute solution (2.4) into equation (2.3) in order to determine k'^2 and k^2 .

Suppose that in equation (2.3) we have $m = n + 2$, $n \geq 1$, and that all deviations $\tilde{\theta}_j$ and $\tilde{\mu}_j$ are multiples of the period $2K$ for $j = \overline{1, n}$, and $\tilde{\mu}_{n+1} = 2K$, $\tilde{\mu}_{n+2} = K$. Then, using $\varphi(u + 2K) = \varphi(u)$, equation (2.3) can be written as

$$\alpha \prod_{j=1}^n \varphi(u + \theta_j) + \beta \prod_{j=1}^m \varphi(u + \mu_j) = \varphi^n(u) [\alpha + \beta \varphi(u) \varphi(u + K)] = 0, \quad (2.8)$$

$$\varphi(u + 2K) = \varphi(u).$$

Since along function (2.4)

$$\varphi(u) \varphi(u + K) = A^2 k', \quad (2.9)$$

it follows that whenever condition

$$\alpha + \beta A^2 k' = 0 \quad (2.10)$$

holds, function (2.4) satisfies equation (2.3), that is, (2.8) is fulfilled.

From relation (2.9) and the formula for A^2 (2.7), we obtain a quadratic equation for the complementary modulus k' , $k'^2 \neq -1$, $k'^2 \neq 1$ (since $k^2 \neq 0$, $k^2 \neq 1$ and $k^2 + k'^2 = 1$):

$$k'^2 - \frac{2a_1\beta}{b_1\alpha} k' + 1 = 0. \quad (2.11)$$

Under the condition imposed on k' , this equation must have two distinct roots:

$$k'_{1,2} = \frac{a_1\beta \pm \sqrt{a_1^2\beta^2 - b_1^2\alpha^2}}{b_1\alpha}, \quad a_1^2\beta^2 \neq b_1^2\alpha^2. \quad (2.12)$$

Substituting the values $k'_{1,2}$ thus found into the formula

$$q^2 = c_0^2 - \frac{a_1}{1 + k'^2}, \quad |q| \neq 1, \quad (2.13)$$

we obtain two pairs of values $q_{1,2}^1$ and $q_{1,2}^2$ corresponding to k'_1 and k'_2 , respectively.

The following conditions must hold:

$$\left| c_0^2 - \frac{a_1}{1+k'^2} \right| \neq 0, \quad \left| c_0^2 - \frac{a_1}{1+k'^2} \right| \neq 1. \quad (2.14)$$

For the roots

$$q_2^1 = -\sqrt{c_0^2 + \frac{a_1}{1+k_1'^2}} = -q_1^1, \quad q_2^2 = -\sqrt{c_0^2 + \frac{a_1}{1+k_2'^2}} = -q_2^1,$$

the functions $\varphi(z - q_1^1 \bar{z})$, $\varphi(z - q_2^1 \bar{z})$ are not analytic (i.e. they do not satisfy the Beltrami equation). Therefore, the solutions are the functions

$$\varphi_{1,2}^1 = \pm \sqrt{\frac{2a_1}{b_1(1+k_1'^2)}} dn[z + q_1^1 \bar{z}], \quad \varphi_{1,2}^2 = \pm \sqrt{\frac{2a_1}{b_1(1+k_2'^2)}} dn[z + q_2^1 \bar{z}]. \quad (2.15)$$

These solutions have periods

$$\tau_1^j = 2K_j = 2K(k_j), \quad \tau_2^j = 4iK'_j = 4iK'(k'_j), \quad j = 1, 2,$$

where

$$k_j^2 = 2 \left(1 - \frac{a_1 \beta}{b_1 \alpha} \right) k'_j, \quad j = 1, 2. \quad (2.16)$$

Of these two solutions, only one solution of the system of equations (2.2) and (2.3) is selected. This choice depends on the values taken by $\tilde{\theta}_j$ and $\tilde{\mu}_j$.

Theorem 2.2. *Let all conditions of Theorem 2.1 be satisfied, $a_1^2 \beta^2 \neq b_1^2 \alpha^2$, $m = n + 2$, and let the complementary moduli $k'_{1,2}$ of the function $dn u$ be computed by formulas (2.12). If $m = n + 2$ and the deviations $\tilde{\theta}_j$, $\tilde{\mu}_j$ are multiples of the period $2K_1$ for $j = 1, 2, \dots, n$, and $\tilde{\mu}_{n+1} = 2K_1$, $\tilde{\mu}_{n+2} = K_1$, then equation (2.3) has solutions of the form*

$$\varphi_{1,2}^1 = \pm i \sqrt{\frac{2a_1}{b_1(1+k_1'^2)}} dn[u, k_1^2], \quad (2.17)$$

where $u = z + q_1^1 \bar{z}$, $k_1^2 = 2 \left(1 - \frac{a_1 \beta}{b_1 \alpha} \right) k'_1$.

The solution of equation (0.4) is now obtained as the solution of the system

$$w_{\bar{z}\bar{z}} - c_0^2 w_{zz} + a_1 w(z) + b_1 w^3(z) = 0, \quad (2.18)$$

$$\alpha \prod_{j=1}^n w(z + \theta_j) + \beta \prod_{j=1}^m w(z + \mu_j) = 0. \quad (2.19)$$

The solution of this system is given by formula (2.1).

Under the conditions of Theorems 2.1 and 2.2, the functions

$$\varphi(u_j) = \varphi(z + q_j \bar{z})$$

have periods τ_1^j , τ_2^j with $\text{Im}(\tau_2^j / \tau_1^j) \neq 0$, $j = 1, 2$. Therefore, the function

$$w(z) = \varphi(u_j) = \varphi(z + q_j \bar{z}), \quad j = 1, 2,$$

for $|q_j| \neq 1$ has periods

$$h_1^j = \frac{1}{1 - |q_j|^2} (\tau_1^j - q_j \bar{\tau}_1^j), \quad h_2^j = \frac{1}{1 - |q_j|^2} (\tau_2^j - q_j \bar{\tau}_2^j), \quad j = 1, 2. \quad (2.20)$$

Consequently, equation (2.18) has doubly-periodic solutions of the form (2.15) with respective periods h_1^1, h_2^1 and h_1^2, h_2^2 computed by formulas (2.16), (2.20). The solution of equation (2.19) with $m = n + 2$ depends on the properties of θ_j, μ_j :

$$\begin{aligned} \theta_j, \mu_j &\in h_1^1 \{2, 4, 6, 8, \dots\}, \quad j = \overline{1, 2}, \\ \mu_{n+1} &= h_1^1, \quad \mu_{n+2} = \frac{1}{2}h_1^1, \end{aligned} \quad (2.21)$$

or

$$\begin{aligned} \theta_j, \mu_j &\in h_1^2 \{1, 2, 3, \dots, n\}, \quad j = 1, 2, 3, \dots, n, \\ \mu_{n+1} &= h_1^2, \quad \mu_{n+2} = \frac{1}{2}h_1^2. \end{aligned} \quad (2.22)$$

If condition (2.21) is satisfied, then equation (2.19) has solutions of the form

$$w(z) = \pm i \sqrt{\frac{2a_1}{b_1(1+k_1'^2)}} dn[z + q_1^1 \bar{z}; k_1^2], \quad (2.23)$$

with periods (h_1^1, h_2^1) , $\text{Im}(h_2^1/h_1^1) \neq 0$, and

$$q_1 = \sqrt{c_0^2 + \frac{a_1}{1+k_1'^2}}, \quad k_1^2 = 2 \left(1 - \frac{a_1\beta}{b_1\alpha}\right) k_1^1. \quad (2.24)$$

The following result holds.

Theorem 2.3. *Let all parameters in equation (0.4) be non-zero, $a_1^2\beta^2 \neq b_1^2\alpha^2$, $m = n + 2$, and let the complementary modulus k'^2 and the modulus k^2 for the delta-amplitude function $dn u = dn(z + q\bar{z})$, $|q| \neq 1$, be computed by formulas (2.12), (2.16). If conditions (2.14) are satisfied and the constant q , $|q| \neq 1$, is computed by the formula*

$$q_1 = + \sqrt{c_0^2 + \frac{a_1}{1+k_1'^2}},$$

and if for the deviations θ_j, μ_j conditions (2.21) hold, then the functions

$$w(z) = \pm i \sqrt{\frac{2a_1}{b_1(1+k_1'^2)}} dn[z + q_1^1 \bar{z}; k_1^2], \quad (2.25)$$

are solutions of equation (0.4) with periods (h_1^1, h_2^1) , where h_1^1, h_2^1 are computed by formulas (2.16), (2.20).

Indeed, by virtue of the foregoing arguments, when the conditions of the theorem are satisfied, function (2.25) simultaneously annihilates both the left-hand and right-hand sides of equation (0.4) by virtue of equations (2.18), (2.19).

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