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MEAN-SQUARED APPROXIMATION OF COMPLEX FUNCTION BY FOURIER SERIES WITH ORTHOGONAL SYSTEM IN THE WEIGHTED BERGMAN SPACE WITH JACOB'S WEIGHT

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Abstract. Let $\mathcal{A}(U)$ is the set of analytic functions f in a unit disk of $U := \{z : |z| < 1\}$; $\mathcal{B}^{(r)}_{2,\gamma}$ and $\mathcal{B}^{(r)}_{2,\gamma^{\beta}_{\alpha}}$, $r \in \mathbb{Z}_+$ are classes of functions $f \in \mathcal{A}(U)$, where $f^{(r)}$, $r \in \mathbb{N}$ belongs to weighted Bergman space correspondingly with a weight γ and Jacobi weight $\gamma^{\beta}_{\alpha} = |z|^{\alpha}(1 - |z|)^{\beta}\, \alpha, \beta > -1$; $\mathcal{H}^{(r)}_{2,\gamma^{\beta}_{\alpha}}$ $(\alpha \geq 0,\ \beta > -1)$ be the class of functions $\mathcal{B}^{(r)}_{2,\gamma^{\beta}_{\alpha}}$, satisfying the condition $\|f^{(r)}\|_{2,\gamma^{\beta}_{\alpha}} \leq 1$. In this paper we study the problem on finding the sharp values of mean squared approximation of functions $f \in \mathcal{B}^{(r)}_{2,\gamma^{\beta}_{\alpha}}$ and their simultaneous derivatives $f^{(s)}(1 \leq s \leq r-1, r \geq 2)$ in the metric of space $B_{2,\gamma^{\beta}_{\alpha}}$. We prove the sharp Jackson-Stechkin type of inequality connecting the best mean squared approximation of $f \in \mathcal{B}^{(r)}_{2,\gamma^{\beta}_{\alpha}}$ and Peetre functional.

1. Introduction. Preliminaries

We study mean-squared approximation of Fourier sums of complex functions f regular in a simply connected domain $\mathcal{D} \subset \mathbb{C}$ and belonging to the weighted Bergman space $B_{2,\gamma} := B_{2,\gamma}(\mathcal{D})$, having finite norm

$$\|f\|_{2,\gamma} := \|f\|_{B_{2,\gamma}} = \left(\frac{1}{2\pi} \iint\limits_{(\mathcal{D})} \gamma(|z|)|f(z)|^2 d\sigma \right)^{1/2},$$

where $\gamma(|z|)$ is positive weight function, $d\sigma$ is the area element and, the integral is in the sense of Lebesgue. In case of $\gamma(|z|) \equiv 1$, $B_{2,\gamma}$ change into ordinary Bergman space (see, for example, [1, p.259]). It is necessary to note, that in case of mean approximation of complex functions in a simply connected domain $\mathcal{D} \subset \mathbb{C}$ with Fourier series orthogonally in $\mathcal{D}$, by system of functions $\{\varphi_k(z)\}_{k=0}^{\infty}$, the problem of seeking the sharp constants in Jackson–Stechkin inequality in $B_{2,\gamma}$ were studied in [2–6].

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We shall give the main facts and definitions formulated in [4]. It is well known that the theory of mean-squared approximation of functions f over a domain $\mathcal{D}\subset\mathbb{C}$ is closely related to the theory of orthogonal functions over the domain. A sequence of functions $\{\varphi_k(z)\}_{k=0}^{\infty}$ is said *orthogonal system* with a weight γ of complex functions over the domain $\mathcal{D}\subset\mathbb{C}$ if

$$\frac{1}{2\pi}\iint\limits_{(\mathcal{D})}\gamma(|z|)\varphi_k(z)\overline{\varphi}_l(z)d\sigma=$$

$$=\begin{cases}0, & k\neq l,\ k,l\in\mathbb{N},\\ \dfrac{1}{2\pi}\iint\limits_{\mathcal{D}}\gamma(|z|)|\varphi_k(z)|^2d\sigma:=\|\varphi_k\|_{2,\gamma}^2, & k=l,\ k\in\mathbb{N}.\end{cases}$$

If

$$\|\varphi_k\|_{2,\gamma}=\left(\frac{1}{2\pi}\iint\limits_{(\mathcal{D})}\gamma(|z|)|\varphi_k(z)|^2d\sigma\right)^{1/2}=1,$$

then such system is called *orthonormal.* Obviously, if the system $\{\varphi_k(z)\}_{k=0}^{\infty}$ is orthogonal, then $\{\varphi_k(z)\cdot\|\varphi_k\|_{2,\gamma}^{-1}\}_{k=0}^{\infty}$ is orthonormal system.

For the function f we associate its Fourier series in this orthogonal system $\{\varphi_k(z)\}_{k=0}^{\infty}$:

$$f(z)=\sum_{k=0}^{\infty}a_k(f)\varphi_k(z), \tag{1.1}$$

where

$$a_k(f)=\frac{1}{2\pi\|\varphi_k\|_{2,\gamma}^2}\iint\limits_{(\mathcal{D})}\gamma(|z|)f(z)\overline{\varphi_k(z)}d\sigma$$

are Fourier coefficients of f. Let

$$S_{n-1}(f,z)=\sum_{k=0}^{n-1}a_k(f)\varphi_k(z) \tag{1.2}$$

be the nth partial sum of the series (1.1). Let us form a linear combination of first n functions of the system $\{\varphi_k(z)\}_{k=0}^{\infty}$:

$$p_{n-1}(z)=\sum_{k=0}^{n-1}b_k\varphi_k(z),\ b_k\in\mathbb{C}, \tag{1.3}$$

and the set of all *generalized polynomials* of form (1.3) we denote by $\mathcal{P}_{n-1}$.

The magnitude

$$E_{n-1}(f)_{2,\gamma}:=\inf\{\|f-p_{n-1}\|_{2,\gamma}:\ p_{n-1}(z)\in\mathcal{P}_{n-1}\}$$

is called the best mean-squared approximation of function $f\in B_{2,\gamma}$ by subspace $\mathcal{P}_{n-1}$.

Consider the case in which $\mathcal{D} \subset \mathbb{C}$ is the unit disk $U := \{z \in \mathbb{C} : |z| \leq 1\}$. In such case, the system of functions $\{z^k\}_{k=0}^{\infty}$ is orthogonal but not orthonormal, since

$$\frac{1}{2\pi} \iint\limits_{(U)} \gamma(|z|) z^k \bar{z}^l d\sigma = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \gamma(\rho) \rho^{k+l+1} e^{i(k-l)t} d\rho dt =$$

$$= \begin{cases} 0, & k \neq l, \ k, l \in \mathbb{Z} \\ \displaystyle\int_0^1 \gamma(\rho) \rho^{2k+1} d\rho := \lambda_k > 0, & k \in \mathbb{Z}. \end{cases}$$

Therefore the system of functions

$$\varphi_k^*(z) := \left\{ \left(\sqrt{\lambda_k} \right)^{-1} \cdot z^k \right\}_{k=0}^{\infty}$$

will be orthonormal system. By $\mathcal{A}(U)$ we denote the class of analytic functions in a disk U. Now let f be an arbitrary function from class $\mathcal{A}(U)$. The Maclaurin series of this function has the form

$$f(z) = \sum_{k=0}^{\infty} c_k(f) z^k. \tag{1.4}$$

The Fourier coefficients $a_k(f)$ of function f and Maclaurin coefficients $c_k(f)$ are connected by

$$a_k(f) = c_k(f) \cdot \sqrt{\lambda_k}, \ k \in \mathbb{Z} \tag{1.5}$$

Thus, Fourier series of function $f \in \mathcal{A}(U)$ in the orthonormal system $\varphi_k^*(z) = \left(\sqrt{\lambda_k}\right)^{-1} z^k$ $(k \in \mathbb{Z}_+)$ with respect to (1.5) takes the form

$$f(z) = \sum_{k=0}^{\infty} a_k(f) \varphi_k^*(z) = \sum_{k=0}^{\infty} \sqrt{\lambda_k} c_k(f) \left(\sqrt{\lambda_k} \right)^{-1} z^k = \sum_{k=0}^{\infty} c_k(f) z^k. \tag{1.6}$$

This series coincides with the Maclaurin series (1.4), the Fourier series $f(z)$ inside the disk U represent an analytic function (1.4) and for which is hold the closedness equation

$$\frac{1}{2\pi} \iint\limits_{(U)} \gamma(|z|) |f(z)|^2 d\sigma = \sum_{k=0}^{\infty} |c_k(f)|^2 \int_0^1 \gamma(\rho) \rho^{2k+1} d\rho = \sum_{k=0}^{\infty} \lambda_k |c_k(f)|^2. \tag{1.7}$$

In particular, if $\gamma(|z|) \equiv 1$ then from (1.7) follows the well-known equality (see, for example, [1, pp.208-209])

$$\|f\|_{2,1}^2 = \|f\|_2^2 = \sum_{k=0}^{\infty} \frac{|c_k(f)|^2}{k+1}.$$

The equality (1.6) means that the Fourier series analytical in a unit disk U, concurrently is Maclaurin series of this function and, according to Weierstrass' theorem [7, p. 107] can be differentiated arbitrary many times and, all differentiated series converge uniformly to the corresponding derivative. For any $r \in \mathbb{N}$ is hold the equality

$$f^{(r)}(z) = \sum_{k=r}^{\infty} \alpha_{k,r} c_k(f) z^{k-r},$$

where for brevity, we set

$$\alpha_{k,r} := k(k-1)\cdots(k-r+1) = k!/(k-r)!,\ k \geq r,\ k, r \in \mathbb{N}.$$

Through $\mathcal{B}_{2,\gamma}^{(r)}$, $r \in \mathbb{Z}_+, r \geq s$ we denote the set of functions $f \in B_{2,\gamma}$ for which $f^{(r)} \in B_{2,\gamma}$, i.e. $\|f^{(r)}\|_{2,\gamma} < \infty$. It is clear that for any $k \geq n > r$, $k, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$

$$f^{(r)}(z) - S_{n-1}^{(r)}(f, z) = f^{(r)}(z) - S_{n-r-1}(f^{(r)}, z) = \sum_{k=n}^{\infty} \alpha_{k,r} c_k(f) z^{k-r},$$

so that we obtain

$$\begin{aligned} E_{n-r-1}^2(f^{(r)})_{2,\gamma} := \inf \left\{ \|f^{(r)} - p_{n-1}^{(r)}\|_{2,\gamma}^2 : p_{n-1}(z) \in \mathcal{P}_{n-1} \right\} = \\ = \|f^{(r)} - S_{n-r-1}(f^{(r)})\|_{2,\gamma}^2 = \sum_{k=n}^{\infty} \alpha_{k,r}^2 |c_k(f)|^2 \int_0^1 \gamma(\rho) \rho^{2(k-r)+1} d\rho. \end{aligned} \tag{1.8}$$

$$\begin{aligned} E_{n-s-1}^2(f^{(s)})_{2,\gamma} = \|f^{(s)} - S_{n-s-1}(f^{(s)})\|_{2,\gamma}^2 = \\ = \sum_{k=n}^{\infty} \alpha_{k,s}^2 |c_k(f)|^2 \int_0^1 \gamma(\rho) \rho^{2(k-s)+1} d\rho. \end{aligned} \tag{1.9}$$

Further, through

$$\mu_s(\gamma) = \int_0^1 \gamma(\rho) \rho^s d\rho,\ s = 0, 1, 2, ... \tag{1.10}$$

we set the moment of order s of weight function $\gamma(\rho)$ on segment $[0, 1]$. According to (1.10), we write (1.8) and (1.9) in the forms

$$E_{n-r-1}^2 \left(f^{(r)}\right)_{2,\gamma} := \sum_{k=n}^{\infty} \alpha_{k,r}^2 |c_k(f)|^2 \mu_{2(k-r)+1}, \tag{1.11}$$

$$E_{n-s-1}^2 \left(f^{(s)}\right)_{2,\gamma} := \sum_{k=n}^{\infty} \alpha_{k,s}^2 |c_k(f)|^2 \mu_{2(k-s)+1}. \tag{1.12}$$

2. The sharp upper bounds of simultaneous approximation

The problem on simultaneous approximation of functions and their derivatives was studied relatively few, while for the best simultaneous polynomial approximation are under development. However, the extreme problems on best simultaneous approximation of smooth functions by spline-functions and their corresponding derivatives was studied by Korneichuk [8]. For the best simultaneous approximation by trigonometric functions was studied in [9] and for analytic functions in a unit disk were studied in [10–12].

In formulating the further results as a weight function, we will consider Jacobi weight

$$\gamma(\rho) := \gamma_\alpha^\beta(\rho) = \rho^\alpha(1-\rho)^\beta, \ \ \alpha, \beta > -1.$$

The following statement is valid

Lemma 2.1. *Let* $k, n \in \mathbb{N}$, $r, s \in \mathbb{Z}_+$, $k \geq n > r \geq s$, $\gamma_\alpha^\beta(\rho) = \rho^\alpha(1-\rho)^\beta$ $\alpha \geq 0$, $\beta > -1$. *Then*

$$\max_{k\geq n>r\geq s} \frac{\alpha_{k,r}^2}{\alpha_{k,r}^2} \cdot \frac{\mu_{2(k-s)+1}(\gamma_\alpha^\beta)}{\mu_{2(k-r)+1}(\gamma_\alpha^\beta)} = \frac{\alpha_{n,s}^2}{\alpha_{n,r}^2} \cdot \frac{\mu_{2(n-s)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}. \tag{2.1}$$

Proof. Obviously, that $\gamma_\alpha^\beta(\rho)$ is continuous for $\alpha, \beta \geq 0$. In this case, it can be easily proved that the equality (2.1) is true. Futhermore, (2.1) is hold for any $\alpha \geq 0, \beta > -1$. Now set

$$\varphi(k, r, s) = \frac{\alpha_{k,r}^2}{\alpha_{k,r}^2} \cdot \frac{\mu_{2(k-s)+1}(\gamma_\alpha^\beta)}{\mu_{2(k-r)+1}(\gamma_\alpha^\beta)}.$$

Let us show that for all $k \geq n > r \geq s$ the function $\varphi(k, r, s)$ is decreasing. Since for $k \geq n > r$

$$\mu_{2(k-r)+1}(\gamma_\alpha^\beta) = \frac{\Gamma(2(k-r)+2+\alpha)\Gamma(\beta+1)}{\Gamma(2(k-r)+3+\alpha+\beta)},$$

$$\mu_{2(k-s)+1}(\gamma_\alpha^\beta) = \frac{\Gamma(2(k-s)+2+\alpha)\Gamma(\beta+1)}{\Gamma(2(k-s)+3+\alpha+\beta)},$$

where $\Gamma(a)$ is Euler's gamma function, then using the formula

$$\Gamma(a+n) = (a+n-1)(a+n-2)\cdots(a+1)\,a\,\Gamma(a)$$

we present $\psi(k, r, s)$ in the form

$$\varphi(k, r, s) = \frac{\alpha_{k,s}^2}{\alpha_{k,r}^2} \cdot \frac{\Gamma(2(k-s)+2+\alpha)}{\Gamma(2(k-s)+3+\alpha+\beta)} \cdot \frac{\Gamma(2(k-r)+3+\alpha+\beta)}{\Gamma(2(k-r)+2+\alpha)}.$$

We make the following ratio

$$\frac{\varphi(k+1, r, s)}{\varphi(k, r, s)} = \frac{\alpha_{k+1,s}^2}{\alpha_{k,s}^2} \cdot \frac{\alpha_{k+1,r}^2}{\alpha_{k+1,r}^2} \cdot \frac{\Gamma(2(k+1-s)+2+\alpha)}{\Gamma(2(k-s)+2+\alpha)} \cdot$$

$$\cdot \frac{\Gamma(2(k-s)+3+\alpha+\beta)}{\Gamma(2(k+1-s)+3+\alpha+\beta)} \cdot \frac{\Gamma(2(k+1-r)+3+\alpha+\beta)}{\Gamma(2(k-r)+3+\alpha+\beta)} \cdot \frac{\Gamma(2(k-r)+2+\alpha)}{\Gamma(2(k+1-r)+2+\alpha)} =$$

$$= \left(\frac{k-r+1}{k-s+1}\right)^2 \cdot \frac{(2(k-s)+2+\alpha)(2(k-s)+3+\alpha)(2(k-r)+3+\alpha+\beta)(2(k-r)+4+\alpha+\beta)}{(2(k-r)+2+\alpha)(2(k-r)+3+\alpha)(2(k-s)+3+\alpha+\beta)(2(k-s)+4+\alpha+\beta)} =$$

$$= \left(1 - \frac{r-s}{k-s+1}\right)^2 \left(1 + \frac{r-s}{k-r+1+\frac{\alpha}{2}}\right) \left(1 + \frac{r-s}{k-r+\frac{3}{2}+\frac{\alpha}{2}}\right) \times$$

$$\times \left(1 - \frac{r-s}{k-s+\frac{3}{2}+\frac{\alpha+\beta}{2}}\right) \left(1 - \frac{r-s}{k-s+2+\frac{\alpha+\beta}{2}}\right) \leq$$

$$\le \left(1-\frac{r-s}{k-s+1}\right)^2\left(1+\frac{r-s}{k-r+1+\dfrac{\alpha}{2}}\right)^2\left(1-\frac{r-s}{k-s+2+\dfrac{\alpha+\beta}{2}}\right)^2=f^2(k,r,s).$$

To accomplish the equality (2.1), it is sufficient to prove that

$$g(k,r,s):=f(k,r,s)-1<0$$

We write

$$g(k,r,s)=\left(1-\frac{r-s}{k-s+1}\right)\left(1+\frac{r-s}{k-r+1+\dfrac{\alpha}{2}}\right)\left(1-\frac{r-s}{k-s+2+\dfrac{\alpha+\beta}{2}}\right)-1=$$

$$g(k,r,s)=-(r-s)\left(\frac{1}{k-s+1}-\frac{1}{k-r+1+\dfrac{\alpha}{2}}+\frac{1}{k-s+2+\dfrac{\alpha+\beta}{2}}\right)-$$

$$-(r-s)^2\left(\frac{1}{(k-s+1)\left(k-r+1+\dfrac{\alpha}{2}\right)}-\frac{1}{(k-s+1)\left(k-s+2+\dfrac{\alpha+\beta}{2}\right)}+\right.$$

$$\left.+\frac{1}{\left(k-r+1+\dfrac{\alpha}{2}\right)\left(k-s+2+\dfrac{\alpha+\beta}{2}\right)}\right)+\frac{(r-s)^3}{(k-s+1)\left(k-r+1+\dfrac{\alpha}{2}\right)\left(k-s+2+\dfrac{\alpha+\beta}{2}\right)}.$$

Further

$$(k-s+1)\left(k-r+1+\frac{\alpha}{2}\right)\left(k-s+2+\frac{\alpha+\beta}{2}\right)\frac{g(k,r,s)}{r-s}=$$

$$=-\left(\left(k-r+1+\frac{\alpha}{2}\right)\left(k-s+2+\frac{\alpha+\beta}{2}\right)-(k-s+1)\left(k-s+2+\frac{\alpha+\beta}{2}\right)+\right.$$

$$\left.+(k-s+1)\left(k-r+1+\frac{\alpha}{2}\right)\right)-(r-s)\left(\left(k-s+2+\frac{\alpha+\beta}{2}\right)-\left(k-r+1+\frac{\alpha}{2}\right)+\right.$$

$$\left.+(k-s+1)\right)+(r-s)^2=-k^2+kr-rs+r-2k-k\alpha+s\frac{\alpha}{2}-\frac{3}{2}\alpha-\frac{\alpha}{2}\cdot\frac{\alpha+\beta}{2}+s+r\frac{\alpha}{2}+ks-1=$$

$$=-(k-1)^2-\alpha\left(k-\frac{r+s}{2}+\frac{6+\alpha+\beta}{4}\right)-(2k+rs-kr-r-s)\le$$

$$\le-(r+2)^2-\alpha\left(r+1-r+\frac{6+\alpha+\beta}{4}\right)-(2r+2+r^2-r^2-r-r-r)=$$

$$=-r^2-4r-4-\alpha\left(1+\frac{6+\alpha+\beta}{4}\right)-2+r=-r^2-3r-6-\alpha\left(1+\frac{6+\alpha+\beta}{4}\right)<0$$

The condition $f(k,r,s)<1$ is hold and $\varphi(k,r,s)<\varphi(n,r,s)$ for $k\ge n>r\ge s$. Lemma is proved. □

Relying on Lemma 1, we state the following theorem.

Theorem 2.2. *For all $k \ge n > r \ge s \ge 1$ and $\alpha \ge 0$, $\beta > -1$ it is hold the equality*

$$\sup_{f \in \mathcal{B}^{(r)}_{2,\gamma^\beta_\alpha}} \frac{E_{n-s-1}(f^{(s)})_{2,\gamma^\beta_\alpha}}{E_{n-r-1}(f^{(r)})_{2,\gamma^\beta_\alpha}} = \frac{\alpha_{n,s}}{\alpha_{n,r}} \cdot \sqrt{\frac{\mu_{2(n-s)+1}(\gamma^\beta_\alpha)}{\mu_{2(n-r)+1}(\gamma^\beta_\alpha)}}. \tag{2.2}$$

Proof. If the function $f \in \mathcal{B}^{(r)}_{2,\gamma^\beta_\alpha}$, then using Lemma 1 and equalities (1.11) and (1.12) we have

$$E^2_{n-s-1}(f^{(s)})_{2,\gamma^\beta_\alpha} = \sum_{k=n}^{\infty} \alpha^2_{k,r} |c_k(f)|^2 \mu_{2(k-r)+1}(\gamma^\beta_\alpha) \left\{ \frac{\alpha^2_{k,s}}{\alpha^2_{k,r}} \cdot \frac{\mu_{2(k-s)+1}(\gamma^\beta_\alpha)}{\mu_{2(k-r)+1}(\gamma^\beta_\alpha)} \right\} \le$$

$$\le \max_{k \ge n > r \ge s} \left\{ \frac{\alpha^2_{k,s}}{\alpha^2_{k,r}} \cdot \frac{\mu_{2(k-s)+1}(\gamma^\beta_\alpha)}{\mu_{2(k-r)+1}(\gamma^\beta_\alpha)} \right\} \cdot \sum_{k=n}^{\infty} \alpha^2_{k,r} |c_k(f)|^2 \mu_{2(k-r)+1}(\gamma^\beta_\alpha) =$$

$$= \frac{\alpha^2_{n,s}}{\alpha^2_{n,r}} \cdot \frac{\mu_{2(n-s)+1}(\gamma^\beta_\alpha)}{\mu_{2(n-r)+1}(\gamma^\beta_\alpha)} \cdot E^2_{n-r-1}(f^{(r)})_{2,\gamma^\beta_\alpha}. \tag{2.3}$$

Since the inequality (2.3) is true for any function $f \in B_{2,\gamma^\beta_\alpha}$, then from (2.3) we obtain upper bound for the left side of equality (2.2)

$$\sup_{f \in \mathcal{B}^{(r)}_{2,\gamma^\beta_\alpha}} \frac{E^2_{n-s-1}(f^{(s)})_{2,\gamma^\beta_\alpha}}{E^2_{n-r-1}(f^{(r)})_{2,\gamma^\beta_\alpha}} \le \frac{\alpha^2_{n,s}}{\alpha^2_{n,r}} \cdot \frac{\mu_{2(n-s)+1}(\gamma^\beta_\alpha)}{\mu_{2(n-r)+1}(\gamma^\beta_\alpha)}. \tag{2.4}$$

To obtain the lower bound of this magnitude, we will consider the function $f_0(z) = z^n \in B_{2,\gamma^\beta_\alpha}$, $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$. For this function from (1.11) and (1.12) it follows at once that

$$E^2_{n-s-1}\left(f_0^{(s)}\right)_{2,\gamma^\beta_\alpha} = \alpha^2_{n,s}\, \mu_{2(n-s)+1}(\gamma^\beta_\alpha), \tag{2.5}$$

$$E^2_{n-r-1}\left(f_0^{(r)}\right)_{2,\gamma^\beta_\alpha} = \alpha^2_{n,r}\, \mu_{2(n-r)+1}(\gamma^\beta_\alpha). \tag{2.6}$$

Given the equalities (2.5) and (2.6), we obtain the lower bound of magnitude (2.2)

$$\sup_{f \in \mathcal{B}^{(r)}_{2,\gamma^\beta_\alpha}} \frac{E^2_{n-s-1}(f)_{2,\gamma^\beta_\alpha}}{E^2_{n-r-1}(f^{(r)})_{2,\gamma^\beta_\alpha}} \ge$$

$$\ge \frac{E^2_{n-s-1}\left(f_0^{(s)}\right)_{2,\gamma^\beta_\alpha}}{E^2_{n-r-1}\left(f_0^{(r)}\right)_{2,\gamma^\beta_\alpha}} = \frac{\alpha^2_{n,s}}{\alpha^2_{n,r}} \cdot \frac{\mu_{2(n-s)+1}(\gamma^\beta_\alpha)}{\mu_{2(n-r)+1}(\gamma^\beta_\alpha)}. \tag{2.7}$$

By comparing the upper bound (2.4) with lower bound (2.7), we get (2.2). This conclude the proof of Theorem 1. □

Corollary 2.3. *[4] Under Theorem 1 condition, for $s = 0$ and all $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n \ge r$ is true the equality*

$$\sup_{f \in \mathcal{B}^{(r)}_2} \frac{E_{n-1}(f)_{2,\gamma^\beta_\alpha}}{E_{n-r-1}(f^{(r)})_{2,\gamma^\beta_\alpha}} = \frac{1}{\alpha_{n,r}} \cdot \sqrt{\frac{\mu_{2n+1}(\gamma^\beta_\alpha)}{\mu_{2(n-r)+1}(\gamma^\beta_\alpha)}}. \tag{2.8}$$

Corollary 2.4. *[5] Under Theorem 1, for $\alpha=\beta=0$, $s=0$ and all $n\in\mathbb{N}$, $r\in\mathbb{Z}_+$, $n\geq r$ is true the equality*

$$\sup_{f\in\mathcal{B}_2^{(r)}}\frac{E_{n-1}(f)_2}{E_{n-r-1}(f^{(r)})_2}=\frac{1}{\alpha_{n,r}}\cdot\sqrt{\frac{n-r+1}{n+1}}. \tag{2.9}$$

Let $\mathcal{H}^{(r)}_{2,\gamma_\alpha^\beta}$ ($\alpha\geq 0$, $\beta>-1$, $r\geq s$) is the set of functions $f\in\mathcal{B}^{(r)}_{2,\gamma_\alpha^\beta}$ for which $\|f^{(r)}\|_{2,\gamma_\alpha^\beta}\leq 1$. We seek the value

$$E_{n-s-1}\left(\mathcal{H}^{(r)}_{2,\gamma_\alpha^\beta}\right)=\sup\left\{E_{n-s-1}(f)_{2,\gamma_\alpha^\beta}\,:\,f\in\mathcal{W}^{(r)}_{2,\gamma_\alpha^\beta}\right\}.$$

Theorem 2.5. *Let $n\in\mathbb{N}$, $r\in\mathbb{Z}_+$, $n>r\geq s$, $\alpha\geq 0$, $\beta>-1$. Then*

$$E_{n-s-1}\left(\mathcal{H}^{(r)}_{2,\gamma_\alpha^\beta}\right)=\frac{\alpha_{n,r}}{\alpha_{n,r}}\cdot\sqrt{\frac{\mu_{2(n-s)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}}. \tag{2.10}$$

Proof. Since $E_{n-r-1}(f^{(r)})_{2,\gamma_\alpha^\beta}\leq\|f^{(r)}\|_{2,\gamma_\alpha^\beta}\leq 1$, then for arbitrary function $f\in\mathcal{H}^{(r)}_{2,\gamma_\alpha^\beta}$ from (2.2) follows that

$$E_{n-s-1}(f)_{2,\gamma_\alpha^\beta}\leq$$
$$\leq\frac{\alpha_{n,s}}{\alpha_{n,r}}\cdot\sqrt{\frac{\mu_{2(n-s)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}}\cdot E_{n-r-1}\left(f^{(r)}\right)_{2,\gamma_\alpha^\beta}\leq\frac{\alpha_{n,s}}{\alpha_{n,r}}\cdot\sqrt{\frac{\mu_{2(n-s)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}},$$

whence passing to upper bounds over functions $f\in\mathcal{H}^{(r)}_{2,\gamma_\alpha^\beta}$ we write

$$E_{n-s-1}\left(\mathcal{H}^{(r)}_{2,\gamma_\alpha^\beta}\right)\leq\frac{\alpha_{n,s}}{\alpha_{n,r}}\cdot\sqrt{\frac{\mu_{2n+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}}. \tag{2.11}$$

On the other hand, for function

$$f_1(z)=\frac{\alpha_{n,s}}{\alpha_{n,r}}\cdot\frac{z^n}{\sqrt{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}},$$

$$E_{n-s-1}(f_1)_{2,\gamma_\alpha^\beta}=\frac{\alpha_{n,s}}{\alpha_{n,r}}\cdot\sqrt{\frac{\mu_{2(n-s)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}}, \tag{2.12}$$

and according to (1.11) we have

$$E_{n-r-1}\left(f_1^{(r)}\right)_{2,\gamma_\alpha^\beta}=1.$$

The last equality means, that $f_1\in\mathcal{H}^{(r)}_{2,\gamma_\alpha^\beta}$, therefore considering (2.12), we write the lower bound estimation

$$E_{n-s-1}\left(\mathcal{H}^{(r)}_{2,\gamma_\alpha^\beta}\right)\geq E_{n-s-1}(f_1)_{2,\gamma_\alpha^\beta}=\frac{\alpha_{n,s}}{\alpha_{n,r}}\cdot\sqrt{\frac{\mu_{2(n-s)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}}. \tag{2.13}$$

The required equality (2.10) is obtained by comparing the upper bound (2.11) and lower bound (2.13). The Theorem 2 is proved. □

3. Peetre $\mathcal{K}$-Functional in $B^{(m)}_{2,\gamma^{\beta}_{\alpha}}$

In this section we shall prove one extreme problem on finding the sharp values of upper bound that is related with the best simultaneous approximation $E_{n-1}(f^{(s)})_{2,\gamma^{\beta}_{\alpha}}$ and Peetre $\mathcal{K}$-functional. The definition and basic properties Peetre $\mathcal{K}$-functional are given in [13,14]. The direct and inverse theorems of the theory of approximation by means of $\mathcal{K}$-functional were proved in [15]. We define the $\mathcal{K}$-functional constructed by the spaces $B_{2,\gamma^{\beta}_{\alpha}}$ and $B^{(m)}_{2,\gamma^{\beta}_{\alpha}}$, $m \in \mathbb{N}$, $\alpha \geq 0$, $\beta > -1$ in the following form

$$\mathcal{K}_m(f;t^m)_{2,\gamma^{\beta}_{\alpha}} := \mathcal{K}\left(f;t^m;B_{2,\gamma^{\beta}_{\alpha}};B^{(m)}_{2,\gamma^{\beta}_{\alpha}}\right) = \\ = \inf\left\{\|f-g\|_{2,\gamma^{\beta}_{\alpha}} + t^m\|g^{(m)}\|_{2,\gamma^{\beta}_{\alpha}} : g \in B^{(m)}_{2,\gamma^{\beta}_{\alpha}}\right\},\ 0<t<1. \tag{3.1}$$

Theorem 3.1. *Let $m, n \in \mathbb{N}$, $r, s \in \mathbb{Z}_+$ and $n > r \geq s$, $n \geq r+m$, $\alpha \geq 0$, $\beta > -1$. Then there is hold the following extreme equality*

$$\sup_{\substack{f\in\mathcal{B}^{(r)}_{2,\gamma}\\ f\notin\mathcal{P}_r}} \frac{(\alpha_{n,r}/\alpha_{n,s})\cdot E_{n-s-1}(f^{(s)})_{2,\gamma^{\beta}_{\alpha}}\left(\mu_{2(n-r)+1}(\gamma^{\beta}_{\alpha})/\mu_{2(n-s)+1}(\gamma^{\beta}_{\alpha})\right)^{\frac{1}{2}}}{\mathcal{K}_m\left(f^{(r)},\ \alpha^{-1}_{n-r,m}\left[\mu_{2(n-r)+1}(\gamma^{\beta}_{\alpha})/\mu_{2(n-r-m)+1}(\gamma^{\beta}_{\alpha})\right]^{\frac{1}{2}}\right)_{2,\gamma^{\beta}_{\alpha}}} = 1. \tag{3.2}$$

Proof. Using the inequality (2.3), for arbitrary function $f \in \mathcal{B}^{(r)}_{2,\gamma^{\beta}_{\alpha}}$, $r \in \mathbb{N}$, we hold

$$E_{n-s-1}(f)_{2,\gamma^{\beta}_{\alpha}} \leq \frac{\alpha_{n,s}}{\alpha_{n,r}}\cdot\left(\frac{\mu_{2(n-s)+1}(\gamma^{\beta}_{\alpha})}{\mu_{2(n-r)+1}(\gamma^{\beta}_{\alpha})}\right)^{\frac{1}{2}}\cdot E_{n-r-1}(f^{(r)})_{2,\gamma^{\beta}_{\alpha}} \leq \\ \leq \frac{\alpha_{n,s}}{\alpha_{n,r}}\cdot\left(\frac{\mu_{2(n-s)+1}(\gamma^{\beta}_{\alpha})}{\mu_{2(n-r)+1}(\gamma^{\beta}_{\alpha})}\right)^{\frac{1}{2}}\cdot\|f^{(r)} - S_{n-r-1}(f)\|_{2,\gamma^{\beta}_{\alpha}}, \tag{3.3}$$

where $S_{n-r-1}(f)$ is a partial sum of $(n-r)$th order of an arbitrary function $g \in B^{(m)}_{2,\gamma^{\beta}_{\alpha}}$. From (1.8) and (2.4) we have

$$\|g - S_{n-r-1}(g)\|_{2,\gamma^{\beta}_{\alpha}} = \\ = E_{n-r-1}(g)_{2,\gamma^{\beta}_{\alpha}} \leq \frac{1}{\alpha_{n-r,m}}\cdot\left(\frac{\mu_{2(n-r)+1}(\gamma^{\beta}_{\alpha})}{\mu_{2(n-r-m)+1}(\gamma^{\beta}_{\alpha})}\right)^{\frac{1}{2}}\cdot\left\|g^{(m)}\right\|_{2,\gamma^{\beta}_{\alpha}}. \tag{3.4}$$

It now follows from inequalities (3.3) and (3.4) that

$$E_{n-s-1}(f)_{2,\gamma^{\beta}_{\alpha}} \leq \frac{\alpha_{n,s}}{\alpha_{n,r}}\cdot\left(\frac{\mu_{2(n-s)+1}(\gamma^{\beta}_{\alpha})}{\mu_{2(n-r)+1}(\gamma^{\beta}_{\alpha})}\right)^{\frac{1}{2}}\cdot\left\{\left\|f^{(r)}-g\right\|_{2,\gamma^{\beta}_{\alpha}} + \|g - S_{n-r-1}(g)\|_{2,\gamma^{\beta}_{\alpha}}\right\} \leq$$

$$\leq \frac{\alpha_{n,s}}{\alpha_{n,r}} \cdot \left(\frac{\mu_{2(n-s)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)} \right)^{\frac{1}{2}} \left\{ \left\| f^{(r)} - g \right\|_{2,\gamma_\alpha^\beta} + \frac{1}{\alpha_{n-r,m}} \cdot \left(\frac{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r-m)+1}(\gamma_\alpha^\beta)} \right)^{\frac{1}{2}} \left\| g^{(m)} \right\|_{2,\gamma_\alpha^\beta} \right\}. \tag{3.5}$$

Since the left side of inequality (3.5) not depend from function $g \in B_{2,\gamma_\alpha^\beta}^{(m)}$ then passing to lower bound over such functions in both parts of (3.5) with respect to the definition of $\mathcal{K}$-functional (3.1) we obtain

$$E_{n-s-1}(f)_{2,\gamma_\alpha^\beta} \leq$$

$$\leq \frac{\alpha_{n,s}}{\alpha_{n,r}} \left(\frac{\mu_{2(n-s)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)} \right)^{\frac{1}{2}} \mathcal{K}_m \left(f^{(r)}, \frac{1}{\alpha_{n-r,m}} \cdot \left[\frac{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r-m)+1}(\gamma_\alpha^\beta)} \right]^{\frac{1}{2}} \right)_{2,\gamma_\alpha^\beta}.$$

From where follows the upper estimation of the left side of (3.4):

$$\sup_{\substack{f \in \mathcal{B}_{2,\gamma_\alpha^\beta}^{(r)} \\ f \notin \mathcal{P}_r}} \frac{(\alpha_{n,r}/\alpha_{n,s}) \cdot E_{n-s-1}(f)_{2,\gamma_\alpha^\beta} \left(\mu_{2(n-r)+1}(\gamma_\alpha^\beta)/\mu_{2(n-s)+1}(\gamma_\alpha^\beta) \right)^{\frac{1}{2}}}{\mathcal{K}_m \left(f^{(r)}, \alpha_{n-r,m}^{-1} \left[\mu_{2(n-r)+1}(\gamma_\alpha^\beta)/\mu_{2(n-r-m)+1}(\gamma_\alpha^\beta) \right]^{\frac{1}{2}} \right)_{2,\gamma_\alpha^\beta}} \leq 1. \tag{3.6}$$

To obtain the lower estimation of the same magnitude, we use that for any arbitrary $p_n \in \mathcal{P}_n$ is hold an inequality (see., for example, [6])

$$\mathcal{K}_m(p_n, t^m)_{2,\gamma_\alpha^\beta} \leq \min \left\{ \|p_n\|_{2,\gamma_\alpha^\beta}, \, t^m \left\| p_n^{(m)} \right\|_{2,\gamma_\alpha^\beta} \right\}. \tag{3.7}$$

We consider the function $f_0(z) = z^n$ for which

$$f_0^{(r+m)}(z) = n(n-1)\cdots(n-r+1)(n-r)\cdots(n-r-m+1)z^{n-r-m} =$$
$$= \alpha_{n,r} \cdot \alpha_{n-r,m} z^{n-(r+m)},$$

then by inequality (3.7) we have

$$\mathcal{K}_m \left(f_0^{(r)}, \frac{1}{\alpha_{n-r,m}} \cdot \left[\frac{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r-m)+1}(\gamma_\alpha^\beta)} \right]^{\frac{1}{2}} \right)_{2,\gamma_\alpha^\beta} \leq$$

$$\frac{1}{\alpha_{n-r,m}} \cdot \left[\frac{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r-m)+1}(\gamma_\alpha^\beta)} \right]^{\frac{1}{2}} \left\| f_0^{(r+m)} \right\|_{2,\gamma_\alpha^\beta} =$$

$$= \frac{1}{\alpha_{n-r,m}} \cdot \left[\frac{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}{\mu_{2(n-r-m)+1}(\gamma_\alpha^\beta)} \right]^{\frac{1}{2}} \alpha_{n,r} \cdot \alpha_{n-r,m} \left[\mu_{2(n-r-m)+1}(\gamma_\alpha^\beta) \right]^{\frac{1}{2}} =$$

$$= \alpha_{n,r} \sqrt{\mu_{2(n-r)+1}(\gamma_\alpha^\beta)}. \tag{3.8}$$

Notice also that

$$E_{n-s-1}\left(f_0^{(s)} \right)_2 = \alpha_{n,s} \sqrt{\mu_{2(n-s)+1}(\gamma_\alpha^\beta)}. \tag{3.9}$$

Using the (3.8) and (3.9) we get:

$$\sup_{\substack{f\in\mathcal{B}^{(r)}_{2,\gamma_\alpha^\beta}\\ f\notin\mathcal{P}_r}} \frac{(\alpha_{n,r}/\alpha_{n,s})\cdot E_{n-s-1}(f)_{2,\gamma_\alpha^\beta}\left(\mu_{2(n-r)+1}(\gamma_\alpha^\beta)/\mu_{2(n-s)+1}(\gamma_\alpha^\beta)\right)^{\frac12}}{\mathcal{K}_m\left(f^{(r)},\,\alpha^{-1}_{n-r,m}\left[\mu_{2(n-r)+1}(\gamma_\alpha^\beta)/\mu_{2(n-r-m)+1}(\gamma_\alpha^\beta)\right]^{\frac12}\right)_{2,\gamma_\alpha^\beta}} \geq$$

$$\geq \sup_{\substack{f\in\mathcal{B}^{(r)}_{2,\gamma_\alpha^\beta}\\ f\notin\mathcal{P}_r}} \frac{(\alpha_{n,r}/\alpha_{n,s})\cdot E_{n-s-1}(f_0)_{2,\gamma_\alpha^\beta}\left(\mu_{2(n-r)+1}(\gamma_\alpha^\beta)/\mu_{2(n-s)+1}(\gamma_\alpha^\beta)\right)^{\frac12}}{\mathcal{K}_m\left(f^{(r)},\,\alpha^{-1}_{n-r,m}\left[\mu_{2(n-r)+1}(\gamma_\alpha^\beta)/\mu_{2(n-r-m)+1}(\gamma_\alpha^\beta)\right]^{\frac12}\right)_{2,\gamma_\alpha^\beta}} = 1 \tag{3.10}$$

The required equality (3.2) we obtain by comparing the upper estimation (3.6) with a lower estimation (3.10). The Theorem 3 is proved. Note that if we put in Theorem 3 $s = 0$ and $\gamma_\alpha^\beta(\rho) \equiv 1$, then we obtain the result which is proved in [16].

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