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INVERSE PROBLEMS OF GEOTERMICS

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Abstract. The solution of the direct problem of geothermia under sedimentation conditions for geothermal reservoirs is considered. The main factors forming the thermal field of sedimentary basins are taken into account in the most complete way — it is the expenditure of heat flow energy on the base for heating of cold sedimentary material, partial shielding of heat flow due to the difference of thermophysical sediments and base rocks, heat generation in accumulating sediments, different rate of sedimentation. The problem of calculating the value of heat flux from the foundation based on temperature observations in wells — the inverse problem of geothermy in sedimentation conditions — has also been solved.

1. Introduction

Geothermal problems can be described by mathematical models, i.e., some set of partial differential equations together with initial and/or boundary conditions defined in a particular domain. Models in computational geothermal quantitatively predict what will happen if the crust and mantle deform slowly over geologic time, often with complications in the form of simultaneous heat transfer (e.g., thermal convection in the mantle), phase changes in the Earth's deep interior, complex rheology (viscosity, plasticity, non-Newtonian fluids), melting and migration of melts, chemical reactions (e.g., thermochemical convection), motion of solid, lateral forces, etc. A mathematical model relates the causal characteristics of a geothermal process to its consequences. The causal characteristics of the simulated process include, for example, the parameters of the initial and boundary conditions, the coefficients, and the right side of the differential equations, as well as geometric parameters, and areas of determination. The purpose of the direct problem is to determine the relationship between the causes and effects of the geothermal process, and therefore to formulate a mathematical problem for a given set of parameters and coefficients. The inverse problem of geothermal is the opposite of the direct problem. The inverse problem is posed when there is no information about the causal characteristics, but there is information about the effects of the geophysical (more specifically, geothermal) process. Inverse problems can be classified as follows: inverse time problems (e.g., to reconstruct the development of a geodynamic process); coefficient problems (e.g., determination

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of coefficients, right sides of model equations), geometric problems (e.g., determination of the location of heat sources in a region or geometry of boundaries), and many others. Inverse problems often turn out to be poorly posed or incorrect in J. Hadamard's terminology [1]. A mathematical model for a geophysical problem should be well-established in the sense that it should have the properties of (1) existence, (2) uniqueness, and (3) stability of the solution of the problem. Tasks for which at least one of these properties is not performed are called poorly defined. If, for example, a problem does not have property (3), then its solution is almost impossible to compute because the calculations are polluted by inevitable errors. If the solution of a problem is not continuously dependent on the initial data, then, generally speaking, the computed solution may have nothing to do with the true solution. In the works of A.N. Tikhonov and his followers, methods for solving incorrect problems are proposed. The essence of A.N. Tikhonov's method is the construction of regularizing families of problems, the solution of which in the limit gives the solution of the initial incorrect problem [2]. The application of A.N. Tikhonov's method to a wide class of geodynamic problems is described in [3].

2. Formulation of the three-dimensional inverse problem of geothermia

Let's $D = \{(x, y, t) : x \in [0, a], y \in [0, b] + t \in [0, t^*]\}$.

The boundaries of area D consist of the following three components

$$\begin{aligned} \Gamma_1 &= \{(x, y, 0) : x(-[0, a]), y[0, b]\}, \\ \Gamma_2 &= \{(0, 0, t) : t \in [0, t^*)\}, \\ \Gamma_3 &= \{(a, b, t) : t \in [0, t^*)\}, \end{aligned}$$

where the initial and boundary data are known.

In the field D we consider the heat conduction equation

$$\frac{\partial u(x, y, t)}{\partial t} = d_1 \frac{\partial^2 u(x, y, t)}{\partial x^2} + d_2 \frac{\partial^2 u(x, y, t)}{\partial y^2}, (x, y, t) \in D, \tag{1}$$

where d_1, d_2-diffusion coefficients, u-temperature, t-time, $u(x, y)$-spatial coordinates, respectively.

On the boundary of Γ_1 the initial condition is set

$$u(x, y, 0) = \varphi(x, y), x \in [0, a], y \in 0, b. \tag{2}$$

On the boundary of Γ_2 the boundary condition is set

$$u(0, 0, t) = \psi(t), t \in [0, t^*]. \tag{3}$$

In the inverse problem we study, the unknown functions are the temperature distribution $U(x, y, t)$ in the region D and the temperature $\theta(t)$ and heat flux $q(t)$ at the boundary Γ_2, for which the Dirichlet and Neumann boundary conditions are satisfied:

$$u(a, b, t) = \theta(t), t \in [0, t^*] \tag{4}$$

$$-k_1 \frac{\partial u(a, b, t)}{\partial x} - k_2 \frac{\partial u(a, b, t)}{\partial y} = q(t), t \in [0, t^*]. \tag{5}$$

The initial mathematical description is augmented with temperature values at some fixed points $x = x_{p_1}, y = y_{p_2}$ where $p \in (0, a), p_2 \in (0, b)$.

If the Green's function $G(x, y, t)$ of the problem is known

$$\frac{\partial G(x,y,t)}{\partial t} = d_1 \frac{\partial^2 G(x,y,t)}{\partial x^2} + d_2 \frac{\partial^2 G(x,y,t)}{\partial y^2}, (x,y,t) \in D,$$

$$G(x,y,0) = 0, G(0,0,t) = 1,$$

$$-k_1 \frac{\partial u(a,b,t)}{\partial x} - k_2 \frac{\partial u(a,b,t)}{\partial y} = 0,$$

then, in accordance with Duhamel's principle, the solution of the problem (1-5) is represented as

$$u(x,y,t) = \int_0^t \frac{\partial G(x,y,t-s)}{\partial t} u(x,y,s)ds, t \in [0,t^*). \tag{6}$$

3. Method of regularization for integral equations of the first kind

The three-dimensional geothermal problem formulated in (2) admits a different formulation using linear integral equations of the first kind

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x-\xi, y-\eta) u(\xi,\eta) d\xi d\eta = f(x,y), \tag{7}$$

where $-\infty < x < \infty, -\infty < y < \infty$, $k(x-\xi, y-\eta) = k(\xi - x, \eta - y) = K(s,t)$-is the symmetric kernel of the equation, $f(x,y)$ is the given function, $u(\xi,\eta)$ is the fast function.

When solving practical problems, integration in (7) is carried out only in finite limits, i.e. we consider Eq.

$$Au \equiv \int_{-a}^{a} \int_{-b}^{b} K(x-\xi, y-\eta) u(\xi,\eta) d\xi d\eta = f(x,y), \tag{8}$$

where $-b \le x \le b, -a \le y \le a$, $A : H \to H$ is a linear integral operator, H is a valid Hilbert space. Naturally, the error of the transition from (7) to (8) must be admissible. Let us assume that the numbers δ_1 and δ_1 characterize the accuracy of the initial data f and the operator A in some chosen metrics. Moreover, for finite limits of integration, we will assume that the function f is known in the rectangle $[-b,\ b] \times [-a, a]$, and the function u outside this region is identically zero, i.e., finite. Then the kernel $K(s,t) = K(x - \xi, y - \eta)$ is defined in the rectangle $[-2b, 2b] \times [-2a, 2a]$, but admits an extension to the plane $R \times R$.

Problem (8) is incorrectly posed [4]. Let us see the regularizing algorithm of its solution based on the method of M.M.Lavrentiev [5] and the fast Fourier transform [6].

Let us consider the case when the operator A is a Hilbert-Schmidt operator, i.e., when the kernel of equation (8) satisfies the condition

$$Au \equiv \int_{-a}^{a} \int_{-b}^{b} \int_{-a}^{a} \int_{-b}^{b} K^2(x-\xi, y-\eta) u(\xi,\eta) d\xi d\eta dx dy < \infty, \tag{9}$$

and the functions $u(\xi,\eta)$ and $f(x,y)$ belong to the two-dimensional Hilbert space $L_2[-a,a;-b,b]$. These conditions (subject to the observance of very non-rigorous for practice constraints [2]) are fulfilled for many geothermal problems reduced to equation (8).

Let's establish some properties of the operator A. The following statement holds.

Theorem 1. *If the function $K(x-\xi,\ y-\eta)$ satisfies the condition (9), then A—compact linear operator in the space $L_2[-a,a;\ -b,\ b]$ and for its norm the following estimates are true*

$$\|A\| \le \Bigg(4ab\int\limits_{-2a}^{2a}\int\limits_{-2b}^{2b}K(s,t)dsdt \ + \ 2a\int\limits_{-2a}^{2a}\int\limits_{-2b}^{2b}sK^2(s,t)dsdt +$$
$$-2b\int\limits_{-2a}^{0}\int\limits_{-2b}^{2b}tK^2(s,t)dsdt \ - \ 2a\int\limits_{-2a}^{2a}\int\limits_{0}^{2b}sK^2(s,t)dsdt -$$
$$-2b\int\limits_{0}^{2a}\int\limits_{-2b}^{2b}tK^2(s,t)dsdt \ + \ \int\limits_{-2a}^{0}\int\limits_{-2b}^{0}stK^2(s,t)dsdt +$$
$$+\int\limits_{0}^{2a}\int\limits_{0}^{2b}stK^2(s,t)dsdt - \int\limits_{-2a}^{0}\int\limits_{0}^{2b}stK^2(s,t)dsdt \ - \ \int\limits_{0}^{2a}\int\limits_{-2b}^{0}stK^2(s,t)dsdt\Bigg)^{1/2} \tag{10}$$

$$\|A\| < 2\Bigg(\int\limits_{-2a}^{2a}\int\limits_{-2b}^{2b}K^2(s,t)dsdt\Bigg)^{1/2}. \tag{11}$$

Theorem 1 is a generalization of a classical result from ([7]) to the case of two-dimensional space L_2.

The norms (10) and (11) generalize to the two-dimensional case of the norm from ([7]).

Let us find the spectrum of the kernel $K(s,t)$, i.e., let us perform for it a twofold Fourier transform of the form

$$k(\omega_1,\omega_2) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}K(s,t)exp[-i(\omega_1 s+\omega_2 t)]dsdt. \tag{12}$$

The corresponding inverse Fourier transform has the form

$$K(s,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}k(\omega_1,\omega_2)exp[-i(\omega_1 s+\omega_2 t)]d\omega_1 d\omega_2. \tag{13}$$

Clearly, if $K(s,t)\in L_2$, then according to Plancherel's theorem ([7]) the spectrum $k(\omega_1,\omega_2)\in L_2$.

Theorem 2. *For the integral operator A of convolution type with a symmetric kernel $K(s,t)$ to be positive in the Hilbert space $L_2[-a,a;-b,b]$ the integral operator*

A of convolution type with a symmetric kernel $K(s,t)$ is positive, it is sufficient that the kernel admits an extension from the region $[-2b,\ 2b]\times[-2a,2a]$ to the whole plane $R\times R$ and the spectrum of the kernel satisfies the condition $0 \le k(\omega_1,\omega_2) < \infty$.

Let's give the scheme of the proof of the theorem. The condition of positivity of the bounded self-adjoint operator A means that

$$(Au,u) \ge 0 \text{ for any } u \in L_2[-a,a;\ -b,\ b]. \tag{14}$$

The boundedness of the operator A follows from estimates (9) and (10). In a real Hilbert space H, the operator A is self-adjoint due to the symmetry of the kernel $K(s,t)$. Let us write in expanded form the scalar product (14)

$$(Au,u) = \int_{-a}^{a}\int_{-b}^{b}\int_{-a}^{a}\int_{-b}^{b} K(x-\xi, y-\eta)u(\xi,\eta)u(x,y)d\xi d\eta dxdy, \tag{15}$$

for any $u \in L_2[-a,a;-b,b]$.

In the expression (15) we substitute the value $K(s,t) = K(x-\xi, y-\eta)$ from the formula (13) and reverse the order of integration. Taking into account that by the condition of Theorem 2 the spectrum of the kernel satisfies the inequality $0 \le k(\omega_1,\omega_2) < \infty$, we obtain

$$K(Au,u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(\omega_1,\omega_2)|\varphi(\omega_1,\omega_2)|^2 d\omega_1 d\omega_2 \ge 0, \tag{16}$$

where

$$\varphi(\omega_1,\omega_2) = \int_{-a}^{a}\int_{-b}^{b} u(x,y)exp[-i(\omega_1 x + \omega_2 y)]dxdy. \tag{17}$$

Theorem 2 is proved. It generalizes to two-dimensional space the corresponding statement from [7].

Note that for positivity of the operator A at $a=\infty$ and $b=\infty$ the following conditions suffice

$$K(s,t) \in L_2[-\infty,\infty;-\infty,\infty] \text{ and } 0 \le k(\omega_1,\omega_2) < \infty.$$

It follows from [6] that in the case of positivity of the operator A, the regularizing solution of the equation (8) is a solution of the following equation

$$\int_{-a}^{a}\int_{-b}^{b} K(x-\xi, y-\eta)u(\xi,\eta)d\xi d\eta + \alpha u(x,y) = f(x,y), \tag{18}$$

where $\alpha = \alpha(\delta_1,\delta_2) > 0$ is the regularization parameter chosen by the nonconvexity method [6].

The solution of the equation (18) is obtained using the Fourier transform. Applying it to both parts of the expression (18) and using the convolution theorem

[6], the function will have the following form

$$\hat{u}(\omega_1, \omega_2, a, b) = \frac{\hat{f}(\omega_1, \omega_2, a, b)}{2\pi(\omega_1, \omega_2) + \alpha}, \tag{19}$$

where $\hat{u}(\omega_1, \omega_2, a, b)$ and $\hat{f}(\omega_1, \omega_2, a, b)$ are Fourier transforms of the functions $u(x, y)$ and $f(x, y)$, respectively, on the region $[-b, b] \times [-a, a]$.

The inverse transformation with respect to (19) gives an approximation to the desired solution

$$u(x, y, a, b) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \frac{f(\omega_1, \omega_2, a, b)}{2\pi k(\omega_1, \omega_2) + \alpha} exp[i(\omega_1 x + \omega_2 y)] d\omega_1 d\omega_2, \tag{20}$$

for which at $\delta_1 \to 0$ and $\delta_2 \to 0$ and increasing limits of integration is true

$$\lim_{\substack{\alpha(\delta_1, \delta_2) \to 0 \\ a \to \infty \\ b \to \infty}} u(x, y, \alpha, \beta) = u(x, y), \tag{21}$$

where $u(x, y)$ is the exact value of the quantity being sought.

For practical realization of calculations by the formula (20), i.e., determination of approximation to the solution, it is most rational to use computational Fourier transform (b. p. F) schemes [8].

Let us write (20) in the form of a two-dimensional inverse discrete Fourier transform [8]:

$$\hat{f}(j_1 \triangle x, j_2 \triangle y) =$$

$$= \frac{1}{2\pi} \sum_{j_2=0}^{N^2-1} \sum_{j_1=0}^{N_2-1} \frac{\hat{f}(p_1 \triangle \omega_1, p_2 \triangle \omega_2) exp[p_1 \triangle \omega_1 j_1 \triangle x + p_2 \triangle \omega_2 j_2 \triangle y] x}{2\pi k(p_1 \triangle \omega_1, p_2 \triangle \omega_2) + \alpha} \triangle \omega_1 \triangle \omega_2 \tag{22}$$

where the expression for $k(p_1 \triangle \omega_1, p_2 \triangle \omega_2)$ is obtained by computing the integral (12) and the two-dimensional discrete Fourier transform

$$\hat{f}(p_1 \triangle \omega_1, p_2 \triangle \omega_2) =$$

$$= \frac{1}{2\pi} \sum_{j_2=0}^{N^2-1} \sum_{j_1=0}^{N_2-1} u(j_1 \triangle x, j_{\triangle} y) exp[-i(p_1 \triangle \omega_1 j_1 \triangle x + p_2 \triangle \omega_2 j_2 \triangle y)] \triangle x, \triangle y$$

$$p_1 = 0.1, ..., N_1 - 1, p_2 = 0.1, ..., N_2 - 1. \tag{23}$$

Assuming that $\triangle x = \triangle y = 1$, then $\triangle \omega_1 = 2\pi/N_1$ and $\triangle \omega_2 = 2\pi/N_2$. Substituting these expressions into (22) and (23), we obtain the final working formulas realized by successive application of one-dimensional B.P.F. algorithms. When solving problems for large arrays, the use of B.P.F. allows reducing the amount of computation by two orders of magnitude compared to direct computation. For example. If $N_1 = 2^{m_1}$ and $N_2 = 2^{m_2}$, where m_1 and m_2 are some natural numbers, then to perform computations (23) using b.p.F. requires approximately $N_1 N_2(m_1 + m_2)$ complex multiplications and additions instead of $N_1 N_2(N_1 + N_2)$ of the same operations in direct computations. Similar questions were analyzed in [9], [10], [11].

4. Conclusion

A further direction of research will be the development of methods for reconstructing land surface temperature from temperature profile measurements in boreholes for glaciers and rocks with constant environmental properties. Surface temperature reconstruction will be proposed in the form of a piecewise constant function and in the form of a segment of trigonometric Fourier series.

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