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THE EXACT WAVE PERIODIC SOLUTIONS FOR SOME GENERALIZED BUSSINESQUE EQUATIONS WITH CONSTANT DEVIATING ARGUMENTS

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Abstract. In this paper, the exact periodic solutions for some generalized Businesque equations with constant deviating arguments are found by the method of decomposition by the elliptic Jacobi function. In the case when not all deviating arguments are multiples of the solution periods, the exact solutions are found using the delta amplitude function $dn\xi$, and the modulus of the function is calculated.

In nonlinear problems of mathematical physics, an important role is played by finding exact solutions of nonlinear wave equations. Recently, a number of methods for finding exact periodic solutions of nonlinear wave equations have been proposed [1-6].

In [2, 4, 6], exact periodic solutions by the $\wp-$ method of the Weierstrass function for the generalized KdF equation and the Kuramato-Sivashensky equation were obtained.

Consider on the plane of variables x, t a fourth order partial derivative equation with deviating arguments

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} + (\alpha_1 - \alpha_2 u(x, t-\tau) u(x,t)) \frac{\partial^3 u}{\partial x^3} - \alpha \frac{\partial^4 u}{\partial x^4} - \\ -\gamma_1 \left(\frac{\partial u}{\partial x}\right)^2 - \gamma_2 u(x+\theta, t) \frac{\partial^2 u}{\partial x^2} = 0, \qquad (1)$$

where $c_0, \alpha_1, \alpha_2, \tau, \alpha, \gamma_1, \gamma_2, \theta-$ are real constants, $u(x,t)-$ is the desired function.

When $\alpha_1 = \alpha_2 = 0,\ \theta = 0,\ \gamma_1 = \gamma_2 = 2\beta$, equation (1) becomes the Businesque equation [1]

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial^4 u}{\partial x^4} - \beta \frac{\partial^2 u^2}{\partial x^2} = 0. \qquad (2)$$

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In [1], the exact wave solution of equation (2) was found by the elliptic Jacobi function expansion method using the functions $sn^2\xi,\ cn^2\xi,\ \xi = k(x-ct)$

$$u(x,t) = \frac{c^2-c_0^2}{2\beta} + \frac{2}{\beta}(1+m^2)\alpha k^2 - \frac{6m^2}{\beta}\alpha k^2 sn^2(k(x-ct)) =$$
$$= \frac{c^2-c_0^2}{2\beta} - \frac{2}{\beta}(2m^2-1)\alpha k^2 + \frac{6m^2}{\beta}\alpha k^2 cn^2(k(x-ct)), \tag{3}$$

here $m^2,\ 0 < m^2 < 1$ the modulus of the elliptic Jacobi function.

Note that the Jacobi functions $sn\xi, cn\xi, dn\xi$ are related by the relations

$$sn^2\xi + cn^2\xi = 1, \quad dn^2\xi + m^2 sn^2\xi = 1 \tag{4}$$

and satisfy, respectively, the differential equations

$$\left(\frac{dsn\xi}{d\xi}\right)^2 = (1-sn^2\xi)(1-m^2sn^2\xi), \tag{5}$$

$$\left(\frac{dcn\xi}{d\xi}\right)^2 = (1-cn^2\xi)(m'^2+m^2cn^2\xi), \tag{6}$$

$$\left(\frac{ddn\xi}{d\xi}\right)^2 = \left(1-dn^2\xi\right)\left(dn^2\xi - m'^2\right), \tag{7}$$

$m'-$ is additive modulus, $m^2 + m'^2 = 1$ and $0 < m < 1,\ 0 < m' < 1$. These functions are bounded

$$-1 \le sn\xi \le 1, \quad -1 \le cn\xi \le 1, \quad m' < dn\xi \le 1.$$

It follows from equations (5)-(7) that the functions $sn\xi,\ cn\xi,\ dn\xi$ are partial solutions of some equation of the Duffing kind [7]

$$\frac{d^2\varphi}{d\xi^2} + a\varphi(\xi) + b\varphi^3(\xi) = 0, \tag{8}$$

when $ab < 0$ or $ab > 0$ and $a, b-$ will be dependent on m^2, m'^2.

The Duffing equation (8) is of importance in nonlinear mechanics, chaos theory, biology, etc. [7].

Exact solutions of the generalized Duffing equation with constant deviations of the argument are studied in [8-11]. The case when the constant deviations of the argument are multiples of the periods of the solution of the equation or are not multiples is investigated. Moreover, interesting results are obtained in the second case. Now replacing in (3) $m^2sn^2\xi$ by $1-dn^2\xi$ by formula (3), we obtain another solution of equation (2)

$$u(x,t) = \frac{c^2-c_0^2}{2\beta} - \frac{2}{\beta}(2-m^2)\alpha k^2 + \frac{6}{\beta}\alpha dn^2[k(x-ct)]. \tag{9}$$

It is known that the modulus $m^2,\ 0 < m^2 < 1$ is an element of the construction of elliptic Jacobi functions [7]. In this paper we will show that when the deviations τ and θ are related to the elliptic integral

$$K(m) = \int_0^{\frac{\pi}{2}} \left(1-m^2sin^2\varphi\right)^{-\frac{1}{2}} d\varphi,$$

the solution of equation (1) can be found using the function $dn\xi$ and the modulus m^2 is exactly calculated using its coefficients.

For this purpose, as in [1], we will look for the solution of equation (1) in the form

$$u(x,t) = \varphi(\xi),\ \xi = x - ct, \tag{10}$$

where the constant $c-$ is the wave parameter.

Substituting (10) into equation (1) for the function $\varphi(\xi)$, we obtain an ordinary differential equation with divergent arguments of the form

$$\left(c^2 - c_0^2\right)\frac{d^2\varphi}{d\xi^2} - \left[\alpha_1 - \alpha_2\varphi(\xi + c\tau)\varphi(\xi)\right]\frac{d^3\varphi}{d\xi^3} - \alpha\frac{d^4\varphi}{d\xi^4} - \\ -\gamma_1\left(\frac{d\varphi}{d\xi}\right)^2 - \gamma_2\varphi(\xi+\theta)\frac{d^2\varphi}{d\xi^2} = 0. \tag{11}$$

If $\varphi(\xi)$ has period $\omega > 0$ and θ is a multiple of period ω, then $\varphi(\xi)$ also satisfies the equation

$$\left(c^2 - c_0^2\right)\frac{d^2\varphi}{d\xi^2} - \left[\alpha_1 - \alpha_2\varphi(\xi + c\tau)\varphi(\xi)\right]\frac{d^3\varphi}{d\xi^3} - \alpha\frac{d^4\varphi}{d\xi^4} - \\ -\gamma_1\left(\frac{d\varphi}{d\xi}\right)^2 - \gamma_2\varphi(\xi)\frac{d^2\varphi}{d\xi^2} = 0. \tag{12}$$

Equations (11) and (12) with respect to the functional relation

$$\varphi(\xi+\omega) = \varphi(\xi),$$

are functionally equivalent.

Now noting that the function $dn\xi$ satisfies the functional relation

$$dn(\xi+K)dn(\xi+2K) = dn(\xi+K)dn\xi = m', \tag{13}$$

where $m'-$ is an additional modulus, $m^2 + m'^2 = 1$, we will look for the solution of equation (11),(12) in the form of

$$\varphi(\xi) = Adn^2\xi = Adn^2\left[\xi, m^2\right], \tag{14}$$

with unknown parameters A, m, m', c.

Taking in equation (11),(12) Taking in equation (11),(12)

$$c\tau = K(m),\ \theta = 2K(m),\ \gamma_1 = \gamma_2 = 2\beta, \tag{15}$$

substitute (14) in equation (12).

From the differential equation formula for the function $dn\xi$, we obtain that the function $\varphi(\xi)$ represented by formula (14) satisfies differential equations of the form

$$\frac{d^2\varphi}{d\xi^2} = -4Am'^2 + 4(1+m'^2)\varphi - \frac{6}{A}\varphi^2,\quad \frac{d^3\varphi}{d\xi^3} = 4(2-m^2)\frac{d\varphi}{d\xi} - \frac{12}{A}\varphi\frac{d\varphi}{d\xi}, \\ \frac{d^4\varphi}{d\xi^4} = 4(2-m^2)\frac{d^2\varphi}{d\xi^2} - \frac{12}{A}\varphi'^2 - \frac{12}{A}\varphi\frac{d\varphi}{d\xi}. \tag{16}$$

Now along a solution of the form (14), we will compare equation (12) with equation (16) when conditions (15) are assumed in equation (12).

Note that if the condition (15) is satisfied, equation (12) will take the form

$$\left(c^2 - c_0^2\right) \frac{d^2\varphi}{d\xi^2} - [\alpha_1 - \alpha_2\varphi(\xi + K)\varphi(\xi)] \frac{d^3\varphi}{d\xi^3} - \alpha \frac{d^4\varphi}{d\xi^4} -$$
$$-2\beta \left(\frac{d\varphi}{d\xi}\right)^2 - 2\beta\varphi\frac{d^2\varphi}{d\xi^2} = 0. \tag{17}$$

Then considering condition (13) and substituting (16) into equation (17), we conclude that a solution of the form (14) satisfies equation (17) if the coefficients of equation (16),(17) are related by the conditions

$$\alpha_1 = \alpha_2 A^2 m'^2, \quad c^2 - c_0^2 = 4(2 - m^2)\alpha, \quad 12\alpha = 2\beta A. \tag{18}$$

From the system of equations (18) at $c \neq c_0$, we find

$$A = \frac{6}{\beta}\alpha, \quad m'^2 = \frac{\alpha_1\beta^2}{36\alpha_2\alpha^2}, \quad c^2 = c_0^2 + 4\alpha + \frac{\alpha_1\beta^2}{9\alpha_2\alpha}. \tag{19}$$

If the condition $\alpha_1\alpha_2 > 0$, $\alpha\beta > 0$ and

$$4\alpha > \beta\sqrt{\frac{\alpha_1}{\alpha_2}}, \tag{20}$$

Substituting the value $m^2 = 1 - m'^2$ into the elliptic integral, we find the period of the function $dn\xi$

$$K = K(m) = \int_0^{\frac{\pi}{2}} \left(1 - \left(1 - \frac{\alpha_1\beta^2}{36\alpha_2\alpha^2}\right) sin^2\varphi\right)^{-\frac{1}{2}} d\varphi.$$

Thus it is true

Theorem 1. *Let in equation (1) $\alpha_1\alpha_2 > 0,\ \gamma_1 = \gamma_2 = 2\beta,\ \alpha\beta > 0$ and condition (20) is satisfied, then if the modulus of the function $dn\xi$ is calculated by the formula*

$$m^2 = 1 - \frac{\alpha_1\beta^2}{36\alpha_2\alpha^2}$$

and constant deviations τ, θ such that

$$\tau c = K(m), \ \theta = 2K(m),$$

then equation (1) admits a solution of the form

$$u(x,t) = \varphi(\xi) = \frac{6}{\beta}\alpha dn^2(x - ct).$$

Now consider an equation of the form

$$\frac{\partial^2 u}{\partial t^2} - c_0^2\frac{\partial^2 u}{\partial x^2} + [\alpha_1 - \alpha_2 u(x, t-\tau)u(x,t)] \frac{\partial^3 u}{\partial x^3} - \alpha\frac{\partial^4 u}{\partial x^4} -$$
$$-\gamma_1 u(x + \theta_1, t) \left(\frac{\partial u}{\partial x}\right)^2 - \gamma_2 u(x + \theta_2, t)u(x,t)\frac{\partial^2 u}{\partial x^2} = 0. \tag{21}$$

It is easy to see that when $\alpha_1 = \alpha_2 = 0,\ \theta_1 = \theta_2 = 0$ and $\gamma_1 = 6\beta,\ \gamma_2 = 3\beta$ this equation takes the form of

$$\frac{\partial^2 u}{\partial t^2} - c_0^2\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial^4 u}{\partial x^4} - 6\beta u\left(\frac{\partial u}{\partial x}\right)^2 - 3\beta u^2\frac{\partial^2 u}{\partial x^2} = 0. \tag{22}$$

or

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial^4 u}{\partial x^4} - \beta \frac{\partial^2 u^3}{\partial x^2} = 0.$$

Equation (22) in traveling wave variables

$$u(x,t) = \varphi(\xi), \ \xi = x - ct, \tag{23}$$

takes the form

$$(c^2 - c_0^2) \frac{d^2\varphi}{d\xi^2} - \alpha \frac{d^4\varphi}{d\xi^4} - 6\beta\varphi \left(\frac{d\varphi}{d\xi}\right)^2 - 3\beta\varphi^2 \frac{d^2\varphi}{d\xi^2} = 0. \tag{24}$$

The solution of this equation will be found in the form

$$\varphi(\xi) = Adn\xi + B = Adn[\xi, m^2] + B, \tag{25}$$

with unknown parameters A, B, c, m^2, m'^2.

Noticing that the function $dn\xi$ satisfies the differential equation

$$\frac{d^2 dn\xi}{d\xi_2} = (2 - m^2) dn\xi - 2dn^3\xi,$$

it can be shown that the function $\varphi(\xi)$, represented by formula (25), satisfies the following differential equations

$$\frac{d^2\varphi}{d\xi^2} = (2 - m^2)(\varphi - B) - \frac{2}{A^2}(\varphi - B)^3, \ \ \frac{d^3\varphi}{d\xi^3} = (2 - m^2)\frac{d\varphi}{d\xi} - \frac{6}{A^2}(\varphi - B)^2 \frac{d\varphi}{d\xi},$$

$$\frac{d^4\varphi}{d\xi^4} = (2 - m^2)\frac{d^2\varphi}{d\xi^2} - \frac{12}{A^2}(\varphi - B)\left(\frac{d\varphi}{d\xi}\right)^2 - \frac{6}{A^2}(\varphi - B)^2 \frac{d^2\varphi}{d\xi^2}. \tag{26}$$

Now along the solution of equation (25), comparing equation (26) with equation (21),(24), we conclude that a function of the form (25) is a solution of equation (24) if the coefficients of equations (24), (26) are related by the conditions

$$c^2 - c_0^2 = \alpha k^2(2 - m^2), \ 12\alpha = 6\beta A^2, \ 6\alpha = 3\beta A^2, B = 0.$$

From this system, we can only find the parameters A and c

$$c^2 = c_0^2 + \alpha(2 - m^2), \ A^2 = \frac{2}{\beta}\alpha.$$

Then equation (22) or (24) at $\alpha\beta > 0$ admits solutions of the form

$$u(x,t) = \varphi(\xi) = \pm\sqrt{\frac{2\alpha}{\beta}} dn(x - ct). \tag{27}$$

We show that by using the solution (27), provided that the deviations τ, θ_1, θ_2 are multiples of the elliptic integral

$$K = K(m), \ \ 0 < m^2 < 1,$$

we can find the solution of equation (21).

If in equation (21) the solution $u(x,t)$ has on the variable x a period $\omega > 0$, and θ_1, θ_2 are multiples of ω and in the traveling wave variables $\xi = x - ct$

$$u(x,t) = \varphi(\xi) = \varphi(x - ct),$$

function $\varphi(\xi)$ has period $\omega > 0$

$$\varphi(\xi+\omega) = \varphi(\xi),$$

then $\varphi(\xi)$ is a solution of the equation

$$\left(c^2 - c_0^2\right) \frac{d^2\varphi}{d\xi^2} + \left[\alpha_1 - \alpha_2\varphi(\xi + kc\tau)\varphi(\xi)\right] \frac{d^3\varphi}{d\xi^3} - \alpha \frac{d^4\varphi}{d\xi^4} -$$
$$-\gamma_1\varphi(\xi)\left(\frac{d\varphi}{d\xi}\right)^2 - \gamma_2\varphi^2(\xi)\frac{d^2\varphi}{d\xi^2} = 0. \tag{28}$$

Taking in this equation $\gamma_1 = 6\beta,\ \gamma_2 = 3\beta$ we will compare it with equation (24) along a function of the form (27). Suppose that the deviations τ, θ_1, θ_2 are related to the elliptic integral $K = K(m)$ by the following conditions

$$c\tau = K,\quad \theta_1 = 2K,\quad \theta_2 = 4K,\quad 0 < m^2 < 1.$$

Then, using the functional equation (13) for $dn\xi$, we conclude that a function of the form (27) satisfies the differential equation (28) if the coefficients of equations (26), (28) are related by the conditions

$$\alpha_1 = \alpha_2 A^2 m',\quad c^2 = c_0^2 + \alpha(1 + m'^2),\quad A^2 = \frac{2}{\beta}\alpha.$$

From this system when $\alpha\beta > 0, \alpha_1\alpha_2 > 0$ and

$$\mid \alpha_1\beta \mid < 2 \mid \alpha_2\alpha \mid, \tag{29}$$

we find

$$m^2 = 1 - \frac{\alpha_1^2\beta^2}{4\alpha_2^2\alpha^2},\quad m'^2 = \frac{\alpha_1^2\beta^2}{4\alpha_2^2\alpha^2}. \tag{30}$$

Hence we have

Theorem 2. *Let $\alpha_1\alpha_2 > 0,\ \gamma_1 = 6\beta,\ \gamma_2 = 3\beta,\ \beta > 0,\ \alpha > 0$ condition (29) be fulfilled in equation (21) and the moduli m^2 and m'^2 for the function $dn\xi$ are calculated by formulas (30).*

Then if the deviations τ, θ_1, θ_2 are such that

$$c\tau = K(m),\ \theta_1 = 2K(m),\ \theta_2 = 4K(m),$$

$$K(m) = \int_0^{\frac{\pi}{2}} \left(1 - \left(1 - \frac{\alpha_1^2\beta^2}{4\alpha_2^2\alpha^2}\right) sin^2\varphi\right)^{-\frac{1}{2}} d\varphi,$$

then equation (21) has solutions of the form

$$u(x,t) = \varphi(\xi) = \pm\sqrt{\frac{2\alpha}{\beta}} dn^2(x - ct),$$

moreover $c = \frac{\theta_1}{2\tau},\ \theta_2 = 2\theta_1$ and

$$\frac{\theta_1^2}{4\tau^2} = c_0^2 + \alpha + \frac{\alpha_1^2\beta^2}{4\alpha_2^2\alpha}.$$

Now consider an equation with several constant deviating arguments of the form

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial^4 u}{\partial x^4} - \beta \frac{\partial^2}{\partial x^2}\left[u(x,t-\tau)u(x,t)\right] =$$
$$= \alpha_1 u(x+\tau_0,t) + \alpha_2 \prod_{j=1}^{3} u(x+\tau_j,t) + \alpha_2 \prod_{j=1}^{5} u(x+\theta_j,t). \tag{31}$$

In traveling wave variables

$$u(x,t) = \varphi(\xi), \quad \xi = x - ct,$$

equation (31) takes the form

$$(c^2 - c_0^2)\frac{d^2\varphi}{d\xi^2} - \alpha \frac{d^4\varphi}{d\xi^4} - \beta \frac{d^2}{d\xi^2}[u(\xi + c\tau)u(\xi)] =$$
$$= \alpha_1 \varphi(\xi+\tau_0) + \alpha_2 \prod_{j=1}^{3} \varphi(\xi+\tau_j) + \alpha_2 \prod_{j=1}^{5} \varphi(\xi+\theta_j). \tag{32}$$

Let the function $\varphi(\xi)$ have period $\omega > 0$. Then if in equation (31) $\tau_0 = \tau_1 = \theta_1$, $c\tau, \tau_2, \theta_2, \theta_2, \theta_3$ are multiples of period ω, then $\varphi(\xi)$ satisfies the equation

$$(c^2 - c_0^2)\frac{d^2\varphi}{d\xi^2} - \alpha \frac{d^4\varphi}{d\xi^4} - \beta \frac{d^2\varphi^2}{d\xi^2} =$$
$$= \varphi(\xi+\tau_0)\left[\alpha_1 + \alpha_2\varphi(\xi)\varphi(\xi+\tau_3) + \alpha_3\varphi^3(\xi)\varphi(\xi+\theta_4)\varphi(\xi+\theta_5)\right]. \tag{33}$$

Equations (32),(33) are functionally equivalent along the periodic function $\varphi(\xi)$

$$\varphi(\xi+\omega) = \varphi(\xi).$$

The solution of equation (33) will be found in the form

$$\varphi(\xi) = Adn^2\xi = Adn^2[\xi; m^2], \tag{34}$$

with unknown parameters A, c, m^2, m'^2.

The function $\varphi(\xi)$ has period $2K$

$$\varphi(\xi + 2K) = \varphi(\xi), \quad K = K(m),$$

and satisfies the functional equation

$$\varphi(\xi)\varphi(\xi+K) = A^2 m'^2. \tag{35}$$

We substitute (34) into equation (33), provided that $c\tau, \tau_2, \theta_3, \theta_4$ are multiples of $2K$ and

$$\tau_3 = K,\ \theta_4 = 3K,\ \theta_5 = 5K$$

and τ_0- is an arbitrary number.

Under these assumptions, comparing equation (33) with equation (32) and considering the functional condition (35), we conclude that the function $\varphi(\xi)$ by formula (34) satisfies equation (33) if the coefficients of equations (33), (2) are related by conditions

$$\alpha_1 + \alpha_2 A^2 m'^2 + \alpha_3 A^4 m'^4 = 0,\ 12\alpha = 2\beta A, \quad c^2 - c_0^2 = 4\alpha(2 - m^2).$$

From this system we determine $A = \frac{6\alpha}{\beta}$ and substituting its value into the first equation we find m'^2 from the equation of degree four

$$\alpha_3 \left(\frac{6\alpha}{\beta}\right)^4 m'^4 + \alpha_2\alpha_3 \left(\frac{6\alpha}{\beta}\right)^2 m'^2 + \alpha_1 = 0. \tag{36}$$

From equation (36), if the conditions $\alpha_2\alpha_3 < 0,\ \alpha_1\alpha_3 > 0$ and

$$\mid \alpha_2 \mid > 2\sqrt{\alpha_1\alpha_3} \tag{37}$$

we find the modulus of m'^2 and m^2 in the form

$$m'^2 = \frac{\beta^2}{36\alpha^2}\left(-\frac{\alpha_2}{2\alpha_3} + \sqrt{\frac{\alpha_2^2}{4\alpha_3^2} - \frac{\alpha_1}{\alpha_3}}\right), \quad m^2 = 1 - m'^2, \tag{38}$$

and the condition must be fulfilled

$$\left|-\frac{\alpha_2}{2\alpha_3} + \sqrt{\frac{\alpha_2^2}{4\alpha_3^2} - \frac{\alpha_1}{\alpha_3}}\right| < \frac{36\alpha^2}{\beta^2}, \tag{39}$$

$$0 < m' < 1,\ 0 < m < 1.$$

Thus it is true

Theorem 3. *Let $\alpha_2\alpha_3 < 0, \alpha_1\alpha_3 > 0$ in equation (31), conditions (37), (39) are satisfied and the moduli m^2, m'^2 of the function $dn\xi$ are calculated by formulas (38). Then, if $\tau_0 = \tau_1 = \theta_1$ and $c\tau, \tau_2, \theta_2, \theta_2, \theta_3$ are multiples of $2K(m)$ and $\tau_3 = K(m), \theta_4 = 3K(m), \theta_5 = 5K(m)$, then equation (31) admits a solution of the following form for any number τ_0*

$$u(x,t) = \frac{6\alpha}{\beta} dn^2(x - ct),$$

and the parameter c is calculated by the formula

$$c^2 = c_0^2 + 4\alpha + 4\alpha m'^2.$$

Consider an equation of the form

$$\frac{\partial^2 u}{\partial t^2} - c_0\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial^4 u}{\partial x^4} - \beta\frac{\partial^2}{\partial x^2}\left[u(x,t-\tau)u^2(x,t)\right] =$$
$$= \alpha_1 u(x+\theta_1,t)\frac{\partial u}{\partial x} + \alpha_2 u(x+\theta_2,t)\frac{\partial u}{\partial t}. \tag{40}$$

This equation in traveling wave variables

$$u(x,t) = \varphi(\xi), \quad \xi = x - ct,$$

will take the form

$$(c^2 - c_0^2)\frac{\partial^2\varphi}{\partial\xi^2} - \alpha\frac{\partial^4\varphi}{\partial\xi^4} - \beta\frac{\partial^2}{\partial\xi^2}\left[\varphi(\xi + c\tau)\varphi^2(\xi)\right] =$$
$$= \alpha_1\varphi(\xi+\theta_1)\frac{\partial\varphi}{\partial\xi} - \alpha_2 c\varphi(\xi+\theta_2)\frac{\partial\varphi}{\partial\xi}. \tag{41}$$

If the function $\varphi(\xi)$ is a solution to this equation and has period $\omega > 0$, and the deviations $c\tau, \theta_1, \theta_2$ are multiples of ω, then $\varphi(\xi)$, also satisfies the differential equation

$$(c^2 - c_0^2)\frac{\partial^2\varphi}{\partial\xi^2} - \alpha\frac{\partial^4\varphi}{\partial\xi^4} - \beta\frac{\partial^2\varphi^3}{\partial\xi^2} = (\alpha_1 - \alpha_2 c)\varphi(\xi)\frac{\partial\varphi}{\partial\xi}. \quad (42)$$

Equations (41) and (42) are functionally equivalent along the periodic function $\varphi(\xi)$ of period $\omega > 0$, under the conditions assumed above.

Finding the solution of equation (42) in the form

$$\varphi(\xi) = Asn\xi = Asn[\xi, m^2], \quad (43)$$

we can easily verify that $\varphi(\xi)$ is the solution of the differential equation

$$\frac{d^4\varphi}{d\xi^4} = (1+m^2)\frac{d^2\varphi}{d\xi^2} + \frac{12m^2}{A^2}\varphi\left(\frac{d\varphi}{d\xi}\right)^2 + \frac{6m^2}{A^2}\varphi^2\frac{d^2\varphi}{d\xi^2}. \quad (44)$$

Now comparing equations (42) and (45) along a solution of the form (44), we conclude that a function of the form (43) is a solution of equation (40) if the coefficients of equation (42) and (44) are related by the conditions

$$\alpha_1 - c\alpha_2 = 0,\ c^2 - c_0^2 = 4\alpha(1+m^2),\ \frac{12m^2\alpha}{A^2} = 6\beta,\ \frac{6m^2\alpha}{A^2} = 3\beta.$$

From this system, when $c = \frac{\alpha_1}{\alpha_2}, \alpha_1^2 \neq c_0^2\alpha_2^2\ \alpha > 0, \beta > 0$ and the condition

$$\alpha + c_0^2 < \frac{\alpha_1^2}{\alpha_2^2} < 2\alpha + c_0^2, \quad (45)$$

find the modulus of m^2

$$m^2 = \frac{1}{\alpha}\left(\frac{\alpha_1^2}{\alpha_2^2} - c_0^2\right) - 1 \quad (46)$$

and

$$A^2 = \frac{2m^2\alpha}{\beta}.$$

Thus it is true

Theorem 4. *Let in the equation (40) $\alpha > 0,\ \beta >,\ \alpha_1\alpha_2 > 0,\ \alpha_1^2 \neq c_0^2\alpha_2^2$ and condition (45) is satisfied and the modulus of the function $sn\xi$, m^2 in calculated by formula (46). Then if the deviations $\tau_1, \theta_1, \theta_2$ are such that $\frac{\alpha_1}{\alpha_2}\tau, \theta_1, \theta_2$ are multiples of $4K(m)$, then equation (40) has solutions of the form*

$$u(x,t) = \pm\sqrt{\frac{2\alpha}{\beta}}msn(x - \frac{\alpha_1}{\alpha_2}t).$$

Note that similar solutions can be obtained using the function $cn\xi$ and $dn\xi$, and with different moduli.

References

1. Liu, S. K., Fu, Z. T., Liu, S.,D., Zhao, Q.: Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, *J. Physics Letters A. – Moscow.* v 289 (2001) 69-74.
2. Porubov, A. V.: Exact travelling wave solutions of nonlinear evolution equation of surface waves in a convicting fluid. *J. Phys. A.: Math. Gen. - Moscow.* v 26 (1993) 797-800.
3. Kudryashov, N. A.: Analytical theory of nonlinear differential equations. *Moscow.* 2004.
4. Kudryashov, N. A.: Exact solutions of the generalized Kuramoto—Sivashinsky equation. *Phys. Lett. A. Moscow.* Vol.147. (1990) 287-291.
5. Kudryashov, N. A.: Simplest equation method to look for exact solutions of nonlinear differential equations. *arXiv:nlin.SI/0406007* v1 4 jun (2004).
6. Safarov, D. S.: About one generalization of the KdF equation.*Ne. equ. mat.fiz. - Kiev.* (1996) 98-101.
7. Sikorsky, Y. S.: Elements of the Theory of Elliptic Functions: with Applications to Mechanics. *M.: KomKniga.* 2006.
8. Moiseev, N. N.: Asymptotic methods in nonlinear mechanics. *M., Nauka* 1981.
9. Safarov, D. S., Miratov, S. K.: On the solution of the generalized Duffing equation with many constant deviations of the argument. *Bulletin of BSU named after Nosir Khusrav. Series of Natural Sciences.* 2/2 (63) (2019) 12-16.
10. Safarov, D. S., Miratov, S. K.: On the solution of one class of second order differential equations with constant deviations of the argument. *Reports of the National Academy of Sciences.* v 64 № 9-10 (2021) 515-525.
11. Safarov, D. S.: Dual-periodic generalized analytic functions and their applications. *Dushanbe, Donish, 2012. - 190 p., 2nd edition Bokhtar* 2021.
12. Safarov, D. S.: Exact bounded periodic solution of the generalized Korteweg-de-Friese (KdF) equation with constant deviating arguments. *Proceedings of the National Academy of Sciences.* v 66, №5-6 (2023) 290-295.
13. Safarov, D. S., Miratov, S. K.: Solution of the Duffing differential equation with three constant deviations of the argument. *Bulletin of BSU named after Nosir Khusrav. Series of Natural Sciences* 2/3(90) (2021) 20 – 24.
14. Safarov, D. S.: On the solution of the Duffing equation with deviating arguments *Proceedings of the Ninth International Thermophysical School, October 6-11, 2014, Tajikistan, Dushanbe - Moscow - Tambov* 2014.
15. Elsholz, L. E., Norkin, S. B.: Introduction to the theory of differential equations with deviating argument. *M., Nauka.* 1971.

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