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TWO-PERIODIC SOLUTIONS OF ONE NONLINEAR ELLIPTIC SYSTEM OF EQUATIONS OF THE FOURTH ORDER IN THE PLANE WITH CONSTANT DEVIATIONS OF THE ARGUMENT

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Abstract. The paper gives an application of generalized elliptic Jacobi functions $sn\omega(z), cn\omega(z), dn\omega(z)$ to finding solutions of one class of nonlinear elliptic systems of equations of the fourth order in the plane with constant deviations of arguments, with Cauchy-Riemann operators $\partial_{\overline{z}}, \partial_z$ and Laplace $\partial^2_{\overline{z}z} = \partial^2_{xx} + \partial^2_{yy}$, where the variables $\overline{z} = x - iy, z = x + iy$ are considered independent variables $2\partial_{\overline{z}} = \partial_x + i\partial_y, 2\partial_{\overline{z}} = \partial_x - i\partial_y$.

Function $\omega(z)-$ quasiperiodic homeomorphism of the Beltrami equation

(0,1) $$\omega_{\overline{z}} - q\omega_z = 0, |q| \neq 1.$$

satisfying the condition

(0,2) $$\omega(0) = 0, \omega(z + h_j) = \omega(z) + \widetilde{h}_j, j = 1, 2,$$

moreover $Im(h_2/h_1) \neq 0$, $Im(\widetilde{h}_2/\widetilde{h}_1) \neq 0$, Function $\omega(z)-$ quasiconformally maps any parallelogram of periods Ω in the plane C_z topped z_0, z_0+h_1, $z_0+h_1+h_2$, z_0+h_2 quadrilaterally Ω' in the plane C_ω topped $\omega(z_0)$, $\omega(z_0)+\tilde{h}_1$, $\omega(z_0)+\tilde{h}_1+\tilde{h}_2$, $\omega(z_0)+\tilde{h}_2$.

The studied equation, on the plane of homeomorphism C_ω, is reduced to a nonlinear ordinary differential equation of the fourth order with constant deviations of the argument. In this case, the solution of the equation is obtained as a bi-periodic solution with one unknown function.

Introduction

In nonlinear equations of mathematical physics an important role is played by nonlinear wave equations having applications in many fields of science and technology, such as fluid mechanics, optical fibers, plasma and elastic media, etc. Therefore, much attention has been paid to finding explicit (exact) traveling wave solutions of these equations. Several methods have been presented to obtain exact solutions for many nonlinear wave equations, such as the Lie-Becklund transform method [8, 21], the inverse problem method of scattering theory [1], the homogeneous balance method [20, 24], hyperbolic tangent expansion method [19, 25, 27], trial function method [10], Hirota bilinear method [9], Weierstrass elliptic function method [7,13, 15], $F-$ function expansion method [23], Jacobi elliptic function expansion method [11, 16, 18], sine - cosine method [26].

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In [7, 14, 17, 22] double-periodic solutions for some classes of elliptic systems of second-order equations with Laplace differential operators $4\partial_{\bar{z}z} = \partial_{\bar{z}z} = \partial_{xx} + \partial_{yy}$ and Bitsadze differential operator $4\partial_{\bar{z}\bar{z}} = 4\partial^2_{\bar{z}} = \partial_{xx} - \partial_{yy} + 2i\partial_{xy}$. This finds explicit formulas between the coefficients of the equation and the modulus of the elliptic Jacobi functions $k^2, k^2 \neq 0, k \neq 1$.

In this case we use differential equations for Jacobi functions on the plane C.

In [7, 12, 13, 14, 15, 17, 28, 29] applications of the method of generalized $\wp-$ Weierstrass function to the solution of some classes of nonlinear elliptic systems of equations of second and third order are given.

In this paper we will find an explicit solution to a nonlinear elliptic system of fourth order equations in complex form [3], [4],[14]

$$w_{zz\overline{zz}}+\alpha_1 w_{\overline{zz}}+\alpha_2 w_{zz}+\alpha_3 w_{\overline{z}}w_z+\alpha_4 w(z+\tau_1)w_{\overline{z}z} = \beta_1 w(z+\tau_2)+\beta_2 w^2(z)w(z+\tau_3), \tag{1.1}$$

where $z = x + iy, \bar{z} = x - iy-$ independent variables, $2\partial_{\overline{z}} = \partial_x + i\partial_y-$ Cauchy-Riemann operator, $2\partial_z = \partial_x - i\partial_y$, $4w_{\overline{zz}} = \partial_{xx} - \partial_{yyy} + 2i\partial_{xy}-$ Bitsadze differential operator, $4w_{zz} = \partial_{xx} - \partial_{yy} - 2i\partial_{xy}, 4w_{z\overline{z}} = \partial_{xx} + \partial_{yy}-$ Laplace operator, $\alpha_j, \beta_i, \tau_k-$ constants, $j = \overline{1,4}, i = \overline{1,2}, k = \overline{1,3}, w = u + i\vartheta-$ the desired function.

We obtain the solution of equation (1.1) as a joint solution of two equations with one unknown function $w(z)$

$$w_{zz\overline{zz}} + \alpha_1 w_{\overline{zz}} + \alpha_2 w_{zz} + \alpha_3 w_{\overline{z}} w_z + \alpha_4 w(z+\tau_1) w_{\overline{z}z} = 0, \tag{1.2}$$

$$\beta_1 w(z+\tau_2) + \beta_2 w^2(z) w(z+\tau_3) = 0, \tag{1.3}$$

1. The method of generalized elliptic functions of Weierstrass and Jacobi

The concept of generalized elliptic function is given in [14], [29].

By a generalized elliptic function we mean a generalized bi-periodic, with periods $h_1, h_2, Im(h_2/h_1) \neq 0$ solution of the Beltrami equation $w(z)$ [4]

$$w_{\bar{z}} - q(z) w_z = 0, \tag{2.1}$$

representable in the form

$$w(z) = \Phi(\omega(z)), \tag{2.2}$$

where $\Phi(\omega(z))$ is a doubly-periodic meromorphic function with periods $\tilde{h}_1, \tilde{h}_2, Im(\tilde{h}_2/\tilde{h}_1) \neq 0$. by ω, $\omega(z)-$ the principal quasiperiodic homeomorphism of equation (1.1) satisfying the condition [12, 14, 29]

$$\omega(0) = 0,\ \omega(z + h_j) = \omega(z) + \tilde{h}_j, \quad j = 1, 2. \tag{2.3}$$

and the constants $\tilde{h}_1, \tilde{h}_2$, as a functional depend on the doubly periodic function $q(z)$ with periods h_1, h_2, satisfying the condition $|q(z)| \leq q_0 < 1,$.

It is shown that at $|q(z)| \leq q_0 < 1$, equation (2.1) has a single one-leaf solution satisfying the condition (2.3) [12, 14, 29].

Using this quasiperiodic homeomorphism, the generalized Weierstrass functions are constructed [12], [29]

$$\tilde{\zeta}(z) = \zeta(\omega(z)), \tilde{\wp}(z) = \wp(\omega(z)), \tilde{\sigma}(z) = \sigma(\omega(z)).$$

We give applications of these functions and the construction of bipartite solutions for a general uniformly elliptic system of first-order equations in the plane [14], [29], and some nonlinear equations of first and second order, [15], [28].

The Weierstrass functions depend on two complex parameters and the periods θ_1, θ_2 can be set arbitrarily with only one general condition $Im(\theta_2, /\theta_1) > 0$.

The Jacobi functions depend on only one complex parameter k, that is, the modulus k^2, which is an element of their construction. And the modulus k^2, is a single-valued function of the parameter $\tau, Im\tau > 0$, and $k^2 \neq 0, k^2 \neq 1$.

In the theory of modular functions [5] it is proved that the equation

$$k^2(\tau) = a,$$

at $a \neq 0, a \neq 1$ has a single solution $\tau, Im\tau > 0$. For each $a \neq 0, a \neq 1$ there is an elliptic function snu, satisfying the differential equation [5]

$$\left(\frac{dsnu}{du}\right)^2 = (1 - sn^2u)(1 - k^2sn^2u), \tag{2.4}$$

$u-$ complex variable.

The functions cnu and dnu are defined by the formulas

$$sn^2u + cn^2 = 1, dn^2u + k^2sn^2u = 1. \tag{2.5}$$

Function $snu-$ bi-periodic with periods $4K$ and $2iK', cnu-$ periodically $4K$ and $2K + 2iK', dnu-$ periodically $2K$ and $4iK'$, where

$$K(k) = \int_0^1 \left[(1-t^2)(1-k^2t^2)\right]^{-1/2} dt,\ K'(k') = \int_0^1 \left[(1-t^2)(1-{k'}^2t^2)\right]^{-1/2} dt.$$

moreover $k^2 \overline{\in} [1, \infty), {k'}^2 \overline{\in} (-\infty, 0]$, [2].

Further, these functions at half-periods satisfy the functional equations [5]

$$snu \cdot sn(u+K') = 1/k',\ cnu \cdot cn(u+K+K') = -ik'/k, dnu \cdot dn(u+K) = k'. \tag{2.6}$$

Second and third order equations with operators $\partial_{\overline{z},\partial_z}$ and constant deviations of the argument are studied in [17], [22], [28]. The solution is found using elliptic Jacobi and Weierstrass functions [2], [5] depending on the fact that some constant deviations are multiples of the period and others are multiples of the half-periods of the solution.

2. Solving the equation (1.2)

When $q-$ is a constant in the Beltrami equation (2.1), its basic quasiperiodic homeomorphism is of the form

$$\omega(z) = z + q\overline{z}, |q| \neq 1, \tag{3.1}$$

and satisfies the conditions $\omega(0) = 0, \omega(z + h_j) = \omega(z) + \theta_j, \theta_j = h_j + q\overline{h}_j, j = 1, 2,$ whereby if $Im(h_2/h_1) \neq 0$, then $Im(\theta_2/\theta_1) \neq 0$.

Biaxial-periodic solutions of equation (1.2) with periods $h_1, h_2, Im(h_2/h_1) \neq 0$, we will look for in the form

$$w(z) = \varphi(\omega) = \varphi(z + q\overline{z}), \tag{3.1}$$

where $q-$ is a constant,$|q|| \neq 1$ and $\varphi(\omega)$ is an analytic function of the variable $\omega = z + q\bar{z}$, that is, $\varphi_{\overline{\omega}} = 0$, and satisfies the Beltrami equation.

For the function $w(z)$ to have periods $h_1, h_2, Im(h_2/h_1) \neq 0$, in formula (3. 1) the function $w(z)$ has periods $h_1, h_2, Im(h_2/h_1) \neq 0$, it is necessary and sufficient that the function $\varphi(\omega)$ be doubly periodic with periods $\theta_j = h_j + q\overline{h}_j, j = 1, 2,$ and it follows from the condition $||q| \neq 1$ that $Im(\theta_2/\theta_1) \neq 0$

Inversely, if the function $\varphi(\omega) = \varphi(z + q\bar{z})$ has periods θ_1, θ_2, then the function $w(z)$ has periods of

$$h_1 = \frac{1}{1 - |q|^2}(\theta_1 - q\bar{\theta}_1), h_2 = \frac{1}{1 - |q|^2}(\theta_2 - q\bar{\theta}_2). \tag{3.2}$$

Substituting (3.1) into (1.2), at $\tau_1 = 0$, for the analytic function $\varphi(\omega)$, we obtain the ordinary differential equation

$$q^2\varphi^{(4)}(\omega) + (\alpha_1 q^2 + \alpha_2)\varphi^{(2)}(\omega) + \alpha_3 q\varphi'^2(\omega) + \alpha_4 q\varphi(\omega)\varphi^2(\omega) = 0, \tag{3.3}$$

When $\alpha_3 = \alpha_4$, this equation is the analog of the Boussinesq equation along the wave solution [6]

$$q^2\varphi^{(4)}(\omega) + (\alpha_1 q^2 + \alpha_2)\varphi^{(2)} + \frac{\alpha_3 q}{2}(\varphi^2(\omega))^{(2)} = 0 \tag{3.4}$$

Following the works [14],[17],[22] we will look for the solution of this equation in the form

$$\varphi(\omega) = Asn^2\omega + B = Asn^2[\omega; k^2] + B, \tag{3.5}$$

where the parameters $A, B, k^2 \neq 0, 1-$ are unknown.

Calculating the derivatives of the function $\varphi(\omega)$ up to the fourth order, by virtue of the differential equation for $sn\omega$, (2.4), we obtain the following differential equations for $\varphi(\omega)$:

$$\varphi'^2 = 4A(\varphi - B) - 4(1 + k^2)(\varphi - B)^2 + \frac{4k^2}{A}(\varphi - B)^3,$$

$$\varphi^{(2)} = 2A - 4(1 + k^2)(\varphi - B) + \frac{6k^2}{A}(\varphi - B)^2,$$

$$\varphi^{(3)} = -4(1 + k^2)\varphi' + \frac{12k^2}{A}(\varphi - B)\varphi',$$

$$\varphi^{(4)} = -4(1 + k^2)\varphi'' + \frac{12k^2}{A}\varphi'^2 + \frac{12k^2}{A}(\varphi - B)\varphi''.$$

Since the derivative of $\varphi^{(4)}$ contains all the lowest terms in equations (3.3), (3.4), with $\alpha_3 = \alpha_4$, it suffices along a solution of the form (3.5) to compare these equations with the equations for $\varphi^{(4)}$. More precisely, with the equations

$$q^2\varphi^{(4)} = -\left(4q^2(1 + k^2) + \frac{12k^2}{A}Bq^2\right)\varphi'' + \frac{12k^2q^2}{A}\varphi'^2 + \frac{12k^2q^2}{A}\varphi\varphi'' \tag{3.6}$$

Comparing equations (3.2), at $\alpha_3 = \alpha_4$, with equation (3.7) along function (3.6), we conclude that if their coefficients are related by the conditions

$$\alpha_1 q^2 + \alpha_2 = 4(1 + k^2)q^2 + \frac{12k^2}{A}Bq^2,$$

$$\alpha_3 q = \alpha_4 q = -\frac{12k^2}{A} B q^2,$$

then function (3.5) satisfies equation (3.4).

From this system we find A and B

$$A = -\frac{12k^2}{\alpha_3} q, \;\; B = -\frac{1}{\alpha_3 q}[\alpha_1 q^2 + \alpha_2 - 4(1+k^2)q^2]. \tag{3.7}$$

Thus at $\alpha_3 = \alpha_4$ equation (3.3) has a solution of the form

$$\varphi(\omega) = -\frac{12k^2}{\alpha_3} q sn^2[\omega, k^2] - \frac{1}{\alpha_3 q}[\alpha_1 q^2 + \alpha_2 - 4(1+k^2)q^2]. \tag{3.8}$$

In this formula, no other conditions are assumed on the constants k^2, q^2 besides the condition $k^2 \neq 0, 1, |q| \neq 1$ yet. The conditions on k^2, q^2 are found after substituting (3.8) into equation (1.3).

In formula (3.8), replacing $sn^2\omega$ by $1 - cn^2\omega$ and $(1 - dn^2\omega)/2$, we obtain more solutions of equation (3.4) in the form of

$$\varphi_1(\omega) = -\frac{12k^2}{\alpha_3} q cn^2\omega - \frac{4(1-2k^2)q}{\alpha_3} - \frac{\alpha_1 q^2 + \alpha_2}{\alpha_3 q}, \tag{3.9}$$

$$\varphi_2(\omega) = -\frac{12}{\alpha_3} q dn^2\omega - \frac{4(2-k^2)q}{\alpha_3} - \frac{\alpha_1 q^2 + \alpha_2}{\alpha_3 q}. \tag{3.10}$$

All functions of the form (3.8),(3.9) give equal rights to the solution of equation (3.4), with periods $2K, 2iK'$. In these formulas the constants $k^2 \neq 0, 1$ and $|q| \neq 1$.

Thus it is true

Theorem 2.1. *Let $\tau_1 = 0, a_3 = a_4$, and $\beta_1 = \beta_2 = 0$. Then if the modulus k^2- of the elliptic functions and the constant q are such that $k^2 \neq 0, 1$ and $|q| \neq 1$, then equation (1.2) admits a biperiodic solution*

$$w(z) = Asn^2(z + q\bar{z}, k^2) + B,$$

where constants A and B are calculated by formulas (3.7), with periods

$$h_1 = \frac{1}{1-|q|^2}(2K - q2\bar{K}), h_2 = \frac{i}{1-|q|^2}(2K' + q2\bar{K}'). \tag{3.11}$$

3. Solution of the functional-difference equation (1.3)

When substituting (3.1), i.e. the formula

$$w(z) = \varphi(\omega) = \varphi(z + q\bar{z}) \tag{4.1}$$

equation (1.3) with respect to the unknown function $\varphi(\omega)$ takes the form

$$\beta_1 \varphi(\omega + \theta_2) + \beta_2 \varphi^2(\omega)\varphi(\omega + \theta_3) = 0, \tag{4.2}$$

where $\theta_2 = \tau_2 + q\bar{\tau}_2, \;\; \theta_3 = \tau_3 + q\bar{\tau}_3$.

If $\beta_1 \neq 0, \beta_2 \neq 0$, then equation (4.2) always has a trivial solution

$$\varphi_0(\omega) = -i\sqrt{\frac{\beta_1}{\beta_2}},$$

and this solution also satisfies equation (3.3)

Thus equation (1.1) always has a trivial, constant solution

$$w(z) = -i\sqrt{\frac{\beta_1}{\beta_2}},$$

for any values of τ_1, τ_2, τ_3.

To obtain a solution to equation (1.1), we must find, along function (3.1), the joint solution of equations (3.2) and (4.1)

Let us show that when in equation (1.1) (or (1.2), (1.3) the constant deviations τ_1, τ_2, τ_3. are related to the periods and half-periods of the functions snu, cnu, dnu, then solutions of equation (4.1) can be obtained using functions of the form (3.8), (3.9) and (3.10) for certain values of q and k^2.

This takes into account the properties (2.6) of the functions snu, cnu, dnu.

Let us show that a function of the form (3.8) as a solution of equation (1.2) at $\tau_1 = 0$ satisfies equation (1.3) or (4.2) if in it $B = 0$, i.e.,

$$\alpha_1 q^2 + \alpha_2 - 4(1 + k^2)q^2 = 0. \tag{4.3}$$

This equation is satisfied e.g. if

$$k^2 = -1 \ \text{ and } \ q^2 = -\frac{\alpha_2}{\alpha_1} \ \text{ and } \ |\alpha_2| \neq |\alpha_1|. \tag{4.4}$$

Then taking into equation (4.2)

$$\theta_2 = 4K \ \text{ and } \ \theta_3 = K'$$

and substituting the function

$$\varphi_0 = -\frac{12k^2}{\alpha_3} q sn^2\omega$$

when $B = 0$ in (4.2) due to the equality

$$sn\omega \cdot sn(\omega + K') = \frac{1}{k'},$$

we get

$$\beta_1 + \beta_2 \frac{144k^4}{\alpha_3^2} q^2 \cdot \frac{1}{k'^2} = 0.$$

Hence we find β_1

$$\beta_1 = -\beta_2 \frac{144}{\alpha_3^2} q^2 \frac{k^4}{k'^2}. \tag{4.5}$$

Thus, under condition (4.5) equation (4.2) has a nontrivial dual-periodic solution of the form

$$\varphi_0(\omega) = -\frac{12k^2}{\alpha_3} q^2 sn^2\omega$$

with periods $2K$ and $2iK'$ if $\theta_2 = 2K$ and $\theta_3 = iK'$.

The following is true

Theorem 3.1. *Let $\beta_1 \neq u\ \beta_2 \neq 0$ and $k^2 \neq 0, k^2 \neq 1, -$ be a complex number modulus function snu and a constant $q, |q|| \neq 1$ such that*

$$\alpha_1 q^2 + \alpha_2 - 4(1 + k^2)q^2 = 0.$$

Then, under condition (4.5) equation (4.3) has a bi-periodic solution of the form

$$w_0(z) = -\frac{12k^2}{\alpha_3} q sn^2(z + q\bar{z}).$$

with periods (3.11)

4. Solving the basic equation (1.1)

Now using the solution of equation (1.2) (Theorem 1) and equation (1.3) (Theorem 2) we obtain the solutions of (1.1)

Theorem 4.1. *Let the complex numbers q, k^2 be such that $|q| \neq 1, k^2 \neq 0, k^2 \neq 1$ and k^2 is the module of the elliptic functions snu cnu dnu and let all the coefficients of equation (1.1) be different from zero and $\alpha_3 = \alpha_4$.*

Let the deviations τ_1, τ_2 be multiples of period h_1 and $\tau_3 = h_2/2$.
Then, if q, k^2 satisfy the equation

$$\alpha_1 q^2 + \alpha_2 + 4(1 + k^2)q^2 = 0,$$

and the condition

$$\beta_1 = -\beta_2 \frac{144k^4}{\alpha_3^2} q^2 \cdot \frac{1}{k'^2}$$

or

$$\beta_1 = -\beta_2 \frac{144 \cdot k^4}{\alpha_3^2(1 - k^2)} q^2,$$

then equation (1.1) has a solution of the form

$$w(z) = \frac{12k^2}{\alpha_3} q sn^2(z + q\bar{z})$$

with periods h_1, h_2- calculated by formula (3.11)

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